Economics 201b
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Solutions to Problem Set 7
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1a. Suppose there is a portfolio z such that $Rz \geq 0$ and $Rz \neq 0$. Then $q \cdot z = \mu \cdot Rz > 0$. If, however, we only have $\mu \geq 0$, then it is possible that the nonzero coordinates of μ and the nonzero coordinates of Rz don't overlap, in which case $q \cdot z = \mu \cdot Rz = 0$.

b. We know that every arbitrage free price q can be represented as $q^T = \mu \cdot R$ for some vector of state multipliers $\mu \geq 0$ (in the previous part, we showed the converse is not true). So suppose there are two arbitrage free prices q_0, q_1 with corresponding vectors of state multipliers μ_0, μ_1, and a portfolio z such that $Rz \geq 0$ and $Rz \neq 0$. Then for any $\alpha \in [0,1]$, the price $q_\alpha = (1-\alpha)q_0 + \alpha q_1$ can be represented as

$$q_\alpha^T = ((1-\alpha)\mu_0 + \alpha\mu_1) \cdot R$$

Then $q_\alpha \cdot z = ((1-\alpha)\mu_0 + \alpha\mu_1) \cdot Rz = (1-\alpha)\mu_0 \cdot Rz + \alpha\mu_1 \cdot Rz > 0$.

c. Define $q = (q_1, q_2, q_3)^T = (4, 5, q_3)$ to be an arbitrage free price. Let $\mu = (\mu_1, \mu_2, \mu_3)^T$ be the corresponding vector of state multipliers. Then

$$q^T = \mu \cdot R \quad \Rightarrow \quad \begin{bmatrix} 4 & 5 & q_3 \end{bmatrix} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \quad \Rightarrow$$

$$4 = \mu_1 + \mu_2 + 3\mu_3 \quad \text{and} \quad 5 = 2\mu_1 + \mu_2 + 4\mu_3 \quad \Rightarrow$$
$$\mu_2 = 1 + 2\mu_1 \quad \text{and} \quad \mu_3 = 1 - \mu_1$$

Thus if we assume the above two equations then

$$\mu \geq 0 \iff \mu_1 \in [0,1]$$

Now if $\mu_1 \in (0,1)$ then $\mu \gg 0$ and the price is arbitrage free by part (a). Thus it suffices to consider the two prices corresponding to $\mu_1 \in \{0,1\}$.

When $\mu_1 = 0$ we have

$$q^T = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 3 \end{bmatrix}$$

Let $z^T = (z_1, z_2, z_3)$ be a portfolio such that $q \cdot z = 0$. Then

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \frac{-4z_1 - 5z_2}{3} \end{bmatrix} \quad \Rightarrow \quad Rz = \begin{bmatrix} -3(z_1+z_2) \\ \frac{-z_1 - 2z_2}{3} \\ \frac{z_1 + 2z_2}{3} \end{bmatrix}$$

Notice that the portfolio $z^T = (-2, 1, 1)$ is way to arbitrage:

$$q \cdot z = \begin{bmatrix} 4 & 5 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 0 \quad \text{and} \quad Rz = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Now consider when $\mu_1 = 1$:

$$q^T = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

Let $z^T = (z_1, z_2, z_3)$ be a portfolio such that $q \cdot z = 0$. Then

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \frac{-4z_1 - 5z_2}{6} \end{bmatrix} \quad \Rightarrow \quad Rz = \begin{bmatrix} -z_1 - \frac{z_2}{2} \\ \frac{2z_1 + z_2}{6} \\ \frac{5z_1 + 7z_2}{3} \end{bmatrix}$$

Notice that the portfolio $z^T = (-1, 2, -1)$ is way to arbitrage:

$$q \cdot z = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = 0 \quad \text{and} \quad Rz = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Thus for q to be arbitrage free, we must have $q_3 \in (3, 6)$.

2a. We know from problem set 1 that in the second time period, with agent utilities of the form $U(x, y) = xy$ and a social endowment of $(a, b) \gg 0$, the set of equilibrium allocations comprise the diagonal line from O_1 to O_2 in the Edgeworth box, and the equilibrium price must be $(\frac{b}{a+b}, \frac{a}{a+b})$. Thus the unique (in $\Delta^o \times \Delta^o$) Radner equilibrium spot price is $p^* = (p_1^*, p_2^*) = ((p_{11}^*, p_{21}^*), (p_{21}^*, p_{22}^*)) = ((\frac{2}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}))$.

b. The return vector for S_1 is $(\frac{2}{3}, \frac{1}{2})^T$ and the return vector for S_2 is $(4, \frac{h}{2})^T$. So the return matrix is

$$\begin{bmatrix} \frac{2}{3} & 4 \\ \frac{1}{2} & \frac{h}{2} \end{bmatrix}$$

It is clear that the matrix has full rank except when $h = 6$. Thus when $h = 6$, p^* is a Hart point. If $h = 0$, then we have

$$R = \begin{bmatrix} R_1 \mid R_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 4 \\ \frac{1}{2} & 0 \end{bmatrix}$$

c. Clearly $(q_1^*, q_2^*) \gg 0$. So fix a price (q, rq) for the two securities where q, $r > 0$. Let w_{si} be the worth (measured in the equilibrium price p_s^*) of agent i's endowment in state s. Agent i's maximization problem can be written as follows

$$\max_{z_i = (z_{1i}, z_{2i})^T,\ x_i = (x_{1i}, x_{2i}) = ((x_{11i}, x_{21i}), (x_{12i}, x_{22i}))} U_i(x_i) \text{ s.t. } \begin{bmatrix} p_1^* \cdot x_{1i} \\ p_2^* \cdot x_{2i} \end{bmatrix} \le \begin{bmatrix} w_{1i} \\ w_{2i} \end{bmatrix} + Rz_i\ , \quad \begin{bmatrix} q \\ rq \end{bmatrix} \cdot z_i \le 0$$

Now we can simplify some of the conditions to make this a tractable maximization problem. All of the budgetary conditions are binding. So

$$z_{2i} = -\frac{z_{1i}}{r}$$

Thus the wealth vector is

$$\begin{bmatrix} w_{1i} \\ w_{2i} \end{bmatrix} + Rz_i = \begin{bmatrix} w_{1i} + z_{1i}[\frac{2}{3} - \frac{4}{r}] \\ w_{2i} + \frac{z_{1i}}{2} \end{bmatrix} \equiv \begin{bmatrix} a_i \\ b_i \end{bmatrix}$$

Let us find the allocation x_{1i} as a function of a_i. From part (a) we know

$$\frac{x_{21i}}{x_{11i}} = 2$$

and the wealth constrain is

$$\frac{2}{3}x_{11i} + \frac{1}{3}x_{21i} = a_i$$

So

$$x_{1i} = (x_{11i}, x_{21i}) = (\frac{3a_i}{4}, \frac{3a_i}{2})$$

Similarly,

$$x_{2i} = (x_{12i}, x_{22i}) = (b_i, b_i)$$

Now since a_i and b_i are functions of z_{1i} we can express the maximization problem purely in terms of z_{1i}:

$$\max_{z_{1i}} \frac{3}{4} \cdot \frac{3}{2}\left(w_{1i} + z_{1i}\left[\frac{2}{3} - \frac{4}{r}\right]\right)^2 + \left(w_{2i} + \frac{z_{1i}}{2}\right)^2$$

taking the derivative and setting equal to zero

$$\frac{9}{4}\left[\frac{2}{3} - \frac{4}{r}\right]\left(w_{1i} + z_{1i}\left[\frac{2}{3} - \frac{4}{r}\right]\right) + \left(w_{2i} + \frac{z_{1i}}{2}\right) = 0 \qquad \Rightarrow$$

$$z_{1i} = \frac{\left(\frac{9}{r} - \frac{3}{2}\right)w_{1i} - w_{2i}}{\left(\frac{3}{2} - \frac{9}{r}\right)\left(\frac{2}{3} - \frac{4}{r}\right) + \frac{1}{2}}$$

Now in equilibrium it must be that

$$z_{11} = -z_{12} \iff (\frac{9}{r} - \frac{3}{2})w_{11} - w_{21} + (\frac{9}{r} - \frac{3}{2})w_{12} - w_{22} = 0 \qquad \Rightarrow$$

$$\left(\frac{9}{r} - \frac{3}{2}\right)(w_{11} + w_{12}) = w_{21} + w_{22} \qquad \Rightarrow$$

$$\left(\frac{9}{r} - \frac{3}{2}\right)\frac{4}{3} = 3 \qquad \Rightarrow$$

$$\frac{9}{r} = \frac{9}{4} + \frac{3}{2} = \frac{15}{4} \qquad \Rightarrow$$

$$r = \frac{q_2^*}{q_1^*} = \frac{12}{5}$$

d. The endowments imply

$$w_{11} = \frac{1}{3} \quad w_{21} = 1 \quad w_{12} = 1 \quad w_{22} = 2$$

Plugging in $r = \frac{12}{5}$ we get

$$z_{11}^* = \frac{\left(\frac{15}{4} - \frac{3}{2}\right)\frac{1}{3} - 1}{\left(\frac{3}{2} - \frac{15}{4}\right)\left(\frac{2}{3} - \frac{5}{3}\right) + \frac{1}{2}} = \frac{-\frac{1}{4}}{\frac{11}{4}} = -\frac{1}{11}$$

$$z_{21}^* = \frac{5}{132}$$

So

$$(z_1^*, z_2^*) = \left(\left(-\frac{1}{11}, \frac{5}{132}\right), \left(\frac{1}{11}, -\frac{5}{132}\right)\right)$$

We can also calculate a_i and b_i for each i:

$$a_1 = \frac{1}{3} - \frac{1}{11}\left(\frac{2}{3} - \frac{5}{3}\right) = \frac{14}{33} \qquad b_1 = 1 - \frac{1}{22} = \frac{21}{22}$$

$$a_2 = 1 + \frac{1}{11}\left(\frac{2}{3} - \frac{5}{3}\right) = \frac{10}{11} \qquad b_2 = 2 + \frac{1}{22} = \frac{45}{22}$$

And finally, we can get the equilibrium allocations

$$x_1^* = \left(\left(\frac{7}{22}, \frac{7}{11}\right), \left(\frac{21}{22}, \frac{21}{22}\right)\right)$$

$$x_2^* = \left(\left(\frac{15}{22}, \frac{15}{11}\right), \left(\frac{45}{22}, \frac{45}{22}\right)\right)$$