Economics 201B–Second Half

Lecture 5, 3/30/10

The Welfare Theorems in the Arrow Debreu Economy

- *Local nonsatiation*: The preference relation $\succeq_i$ on the consumption set X_i is locally nonsatiated if, for every $x_i \in X_i$ and every $\varepsilon > 0$, there exists $x_i' \in X_i$ such that $|x_i' - x_i| < \varepsilon$ and $x_i' \succ_i x_i$.
  - Note that this is a substantial weakening of monotonicity
  - Important especially with production, since we want to allow for input goods which provide no direct consumption utility

Theorem 1 (First Welfare Theorem) *If preferences are locally nonsatiated and (x^*, y^*, p^*, T) is a Walrasian Equilibrium with Transfers, then (x^*, y^*) is Pareto Optimal.*

Proof: Let

$$W_i = p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} p^* \cdot y_j^* + T_i$$

W_i is the income available to person i. Observe that

$$\begin{aligned}
\sum_{i=1}^{I} W_i &= \sum_{i=1}^{I} p^* \cdot \omega_i + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} p^* \cdot y_j^* + \sum_{i=1}^{I} T_i \\
&= p^* \cdot \left(\sum_{i=1}^{I} \omega_i \right) + \sum_{j=1}^{J} \left(\sum_{i=1}^{I} \theta_{ij} \right) p^* \cdot y_j^* + 0 \\
&= p^* \cdot \bar{\omega} + \sum_{j=1}^{J} p^* \cdot y_j^*
\end{aligned}$$

By the definition of Walrasian Equilibrium with Transfers, (x^*, y^*) is a feasible allocation.

Suppose (x^*, y^*) is not Pareto Optimal. Then there is a feasible allocation (x', y') such that

$$
\begin{aligned}
& x'_i \succeq_i x^*_i \quad (i = 1, \ldots, I) \\
& x'_i \succ_i x^*_i \quad \text{for some } i, \text{ WLOG } i = 1
\end{aligned}
$$

$$
\begin{aligned}
x'_1 \succ_1 x^*_1 \quad & \Rightarrow \quad x'_1 \notin B_1(p^*, y^*, T) \\
& \Rightarrow \quad p^* \cdot x'_1 > W_1
\end{aligned}
$$

We claim that

$$p^* \cdot x'_i \geq W_i \ (i = 1, \ldots, I)$$

If not, $p^* \cdot x'_i < W_i$ for some i. The dot product is continuous, so we may find some $\varepsilon > 0$ such that

$$z \in X_i, \ |z - x'_i| < \varepsilon \Rightarrow p^* \cdot z < W_i \tag{1}$$

By local nonsatiation, we may find $x''_i \in X_i$ such that

$$|x''_i - x'_i| < \varepsilon \text{ and } x''_i \succ_i x'_i \text{ so } x''_i \succ_i x^*_i$$

by completeness and transititivity of preferences. Since $|x''_i - x'_i| < \varepsilon$, $p^* \cdot x''_i < W_i$ by Equation (1), so x''_i lies in agent i's budget set, so $x^*_i \notin D_i(p^*, y^*, T)$, a contradiction which shows that $p^* \cdot x'_i \geq W_i$ for all i. Therefore,

$$
\begin{aligned}
p^* \cdot \sum_{i=1}^{I} x'_i \quad &= \quad \sum_{i=1}^{I} p^* \cdot x'_i \\
&> \quad \sum_{i=1}^{I} W_i \\
&= \quad p^* \cdot \bar{\omega} + \sum_{j=1}^{J} p^* \cdot y^*_j
\end{aligned}
$$

$$
\begin{aligned}
&\geq \; p^* \cdot \bar{\omega} + \sum_{j=1}^{J} p^* \cdot y'_j \text{ (profit maximization)} \\
&= \; p^* \cdot \left(\bar{\omega} + \sum_{j=1}^{J} y'_j \right) \\
&= \; p^* \cdot \sum_{i=1}^{I} x'_i \text{ because } (x', y') \text{ is a feasible allocation}
\end{aligned}
$$

so

$$p^* \cdot \sum_{i=1}^{I} x'_i > p^* \cdot \sum_{i=1}^{I} x'_i$$

a contradiction which proves that (x^*, y^*) is Pareto optimal. Another way to interpret the contradiction is that we have shown that (x', y') cannot be a feasible allocation.

■

- *Note*: As in the Edgeworth Box case, if we were content to prove weak Pareto optimality, we wouldn't need to assume local nonsatiation or completeness or transitivity. Weak Pareto Optimality follows directly from the definition of Walrasian Equilibrium with Transfers.

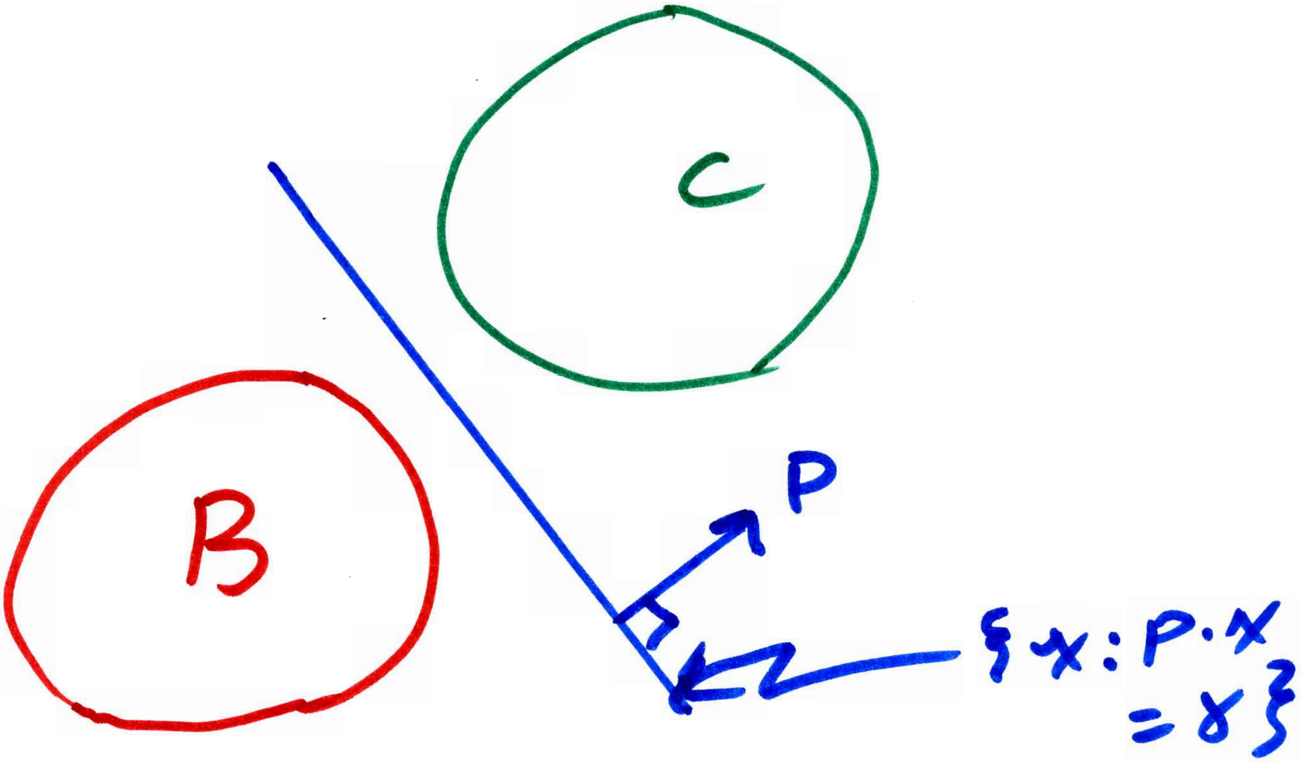
Run-Up to Second Welfare Theorem:

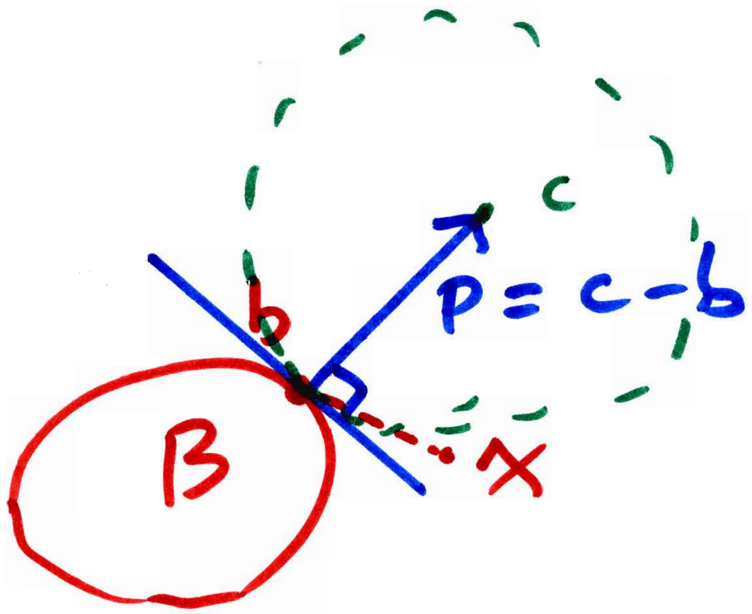
- *supremum, infimum*: sup and inf were covered in 204, review in de la Fuente. Recall

$$\sup B \leq \alpha \quad \Rightarrow \quad b \leq \alpha \text{ for all } b \in B$$

$$\inf B \geq \alpha \quad \Rightarrow \quad b \geq \alpha \text{ for all } b \in B$$

Recall from Econ 204 the following important theorem:


C
B
p
{x : p·x = γ}


c
p = c − b
b
B
x

Theorem 2 (Minkowski's Theorem) (Separating Hyperplane Theorem) *If $B, C \subseteq \mathbf{R}^L$ are convex, $B \neq \emptyset \neq C$, and $B \cap C = \emptyset$, then there exists $p \in \mathbf{R}^L$, $p \neq 0$ such that*

$$\sup p \cdot B \leq \inf p \cdot C$$

In particular, if $b \in B$ and $c \in C$, then $p \cdot b \leq p \cdot c$.

Review rough idea of proof from 204.

Pure Exchange Economy (depart slightly from MWG): A pure exchange economy is an Arrow-Debreu economy in which

- $\bar{\omega} \gg 0$;
- $J = 0$ (no firms, $\sum_{j=1}^{0} Y_j = \{0\}$ (the empty sum is zero)); the only economic activities are trade and consumption;
- $X_i = \mathbf{R}_+^L$, preferences are complete, transitive and locally nonsatiated.

Recall that in the Arrow-Debreu economy, a feasible allocation is (x, y) with

$$\sum_{i=1}^{I} x_i = \bar{\omega} + \sum_{j=1}^{J} y_j$$

but because $J = 0$, $\sum_{j=1}^{J} y_j = 0$, so

$$\sum_{i=1}^{I} x_i = \bar{\omega}$$

i.e. x is an exact allocation. Hence, we can eliminate y, speak of an exact allocation x, and a Walrasian equilibrium (p^*, x^*), or a Walrasian equilibrium with transfers (p^*, x^*, T).

Quasi-Demand: In pure exchange economy, define

$$Q_i(p, T) = \{x_i \in B_i(p, T) : x_i' \succ_i x_i \Rightarrow p \cdot x_i' \geq p \cdot \omega_i + T_i\}$$

Anything strictly preferred uses up whole budget; motivation is limited, but it's technically very convenient.

Example: Consider the utility function $u_1(x) = x_1 + x_2$ and endowment $\omega_1 = (0, 1)$. If $p = (1, 0)$, then

$$\begin{aligned} B_1(p,0) &= \{x \in \mathbf{R}^2_+ : p \cdot x \leq p \cdot \omega_1\} \\ &= \{x \in \mathbf{R}^2_+ : p \cdot x \leq 0\} \\ &= \{(0, x_2) : x_2 \geq 0\} \\ D_1(p,0) &= \emptyset \\ Q_1(p,0) &= B_1(p,0) = \{(0, x_2) : x_2 \geq 0\} \end{aligned}$$

If $u_2(x) = x_1 + x_2$ and endowment $\omega_2 = (1, 0)$, then

$$\begin{aligned} B_2(p,0) &= \{x \in \mathbf{R}^2_+ : p \cdot x \leq p \cdot \omega_2\} \\ &= \{x \in \mathbf{R}^2_+ : p \cdot x \leq 1\} \\ &= \{(x_1, x_2) : x_1 \in [0,1], x_2 \geq 0\} \\ D_2(p,0) &= \emptyset \\ Q_2(p,0) &= \emptyset \end{aligned}$$

Theorem 3 (Second Welfare Theorem) (Pure Exchange Case) *If x^* is Pareto Optimal in a pure exchange economy, with strongly monotone, continuous, convex preferences, there exists a price vector p^* and an income transfer T such that (p^*, x^*, T) is a Walrasian Equilibrium with Transfers.*

Outline of Proof:

- Let

$$\begin{aligned} A_i &= \{x'_i - x^*_i : x'_i \succ_i x^*_i\} \\ A &= \sum_{i=1}^{I} A_i = \{a_1 + \cdots + a_I : a_i \in A_i\} \end{aligned}$$

  Then $0 \notin A$ (if it were, we'd have a Pareto improvement).

- By Minkowski's Theorem, find $p^* \neq 0$ such that

$$\inf p^* \cdot A \geq 0$$

- Show $\left(\mathbf{R}^L_+ \setminus \{0\}\right) \subset A_i$ and hence $p^* \geq 0$.

- Show $\inf p^* \cdot A_i = 0$ for each i.

- Define T to make x_i^* affordable at p^*:

$$T_i = p^* \cdot x_i^* - p^* \cdot \omega_i$$

  Show $\sum_{i=1}^I T_i = 0$ and

$$x_i^* \in Q_i(p^*, T)$$

- Use strong monotonicity to show that $p^* \gg 0$.

- Show

$$p^* \gg 0 \Rightarrow Q_i(p^*, T) = D_i(p^*, T)$$