Bus Ad 239B–Spring 2003
Lecture Notes Part II

10 Detrending and Changing the Variance of Itô Processes (Continued)

2. **Converting a Martingale to a Wiener Process: Time Change** Suppose we want to convert a martingale to a Wiener process. The most natural way to do this is to use a time change.

   Let Z be a (one-dimensional) martingale with respect to the filtration generated by a standard K-dimensional Wiener process W; for simplicity, we consider the case $K = 1$. By the Martingale Representation Theorem, there exists $b \in \mathcal{L}^2$ such that

$$Z(\omega, t) = Z(\omega, 0) + \int_0^t b(\omega, s)\, dW$$

   The instantaneous variance of $Z(\omega, t)$ is $b(\omega, t)^2$. If $|b(\omega, t)| > 1$, Z is moving "too fast" to be a Wiener process, while if $|b(\omega, t)| < 1$, Z is moving "too slow" to be a Wiener process. However, we can fix this by speeding up or slowing down time to make the instantaneous variance equal 1.

   Replace W with the n-step random walk X, and suppose that b is an adapted, simple process, and[1]

$$\frac{\max\left\{b\left(\omega, \frac{k}{n}\right)^2\right\}}{n} \to 0 \text{ as } n \to \infty \tag{1}$$

   Let

$$\tau\left(\omega, \frac{k}{n}\right) = \sum_{j=0}^{k-1} \frac{b\left(\omega, \frac{j}{n}\right)^2}{n}$$

[1] If, in the continuous time setting, $b \in \mathcal{L}^2$, the simple functions approximating it will satisfy the condition in Equation (1) almost surely.

$\tau(\omega,\cdot)$ is nondecreasing in time; if $b(\omega,t) \neq 0$ everywhere, then $\tau(\omega,\cdot)$ is strictly increasing in time. Define

$$\tau^{-1}(\omega,t) = \min\{s : \tau(\omega,s) \geq t\}$$

Then

$$t \leq \tau\left(\omega,\tau^{-1}(\omega,t)\right) \leq t + \frac{\max\left\{b\left(\omega,\frac{k}{n}\right)^2\right\}}{n}$$

so

$$\tau\left(\omega,\tau^{-1}(\omega,t)\right) \simeq t$$

Then

$$\int_0^{\tau^{-1}(\omega,t)} b\,dX \simeq \sum_{k=0}^{\lfloor n\tau^{-1}(\omega,t)\rfloor-1} \frac{b\left(\omega,\frac{k}{n}\right)\omega_{k+1}}{\sqrt{n}}$$

which is a sum of $\lfloor n\tau^{-1}(\omega,t)\rfloor$ uncorrelated random variables. so

$$\begin{aligned}
&\operatorname{Var}\left(\sum_{k=0}^{\lfloor n\tau^{-1}(\omega,t)\rfloor-1} \frac{b\left(\omega,\frac{k}{n}\right)\omega_{k+1}}{\sqrt{n}}\right) \\
&= \sum_{k=0}^{\lfloor n\tau^{-1}(\omega,t)\rfloor-1} \operatorname{Var}\left(\frac{b\left(\omega,\frac{k}{n}\right)\omega_{k+1}}{\sqrt{n}}\right) \\
&= \sum_{k=0}^{\lfloor n\tau^{-1}(\omega,t)\rfloor-1} \left(\frac{b\left(\omega,\frac{k}{n}\right)^2}{n}\right) \\
&= \tau\left(\omega,\frac{\lfloor n\tau^{-1}(\omega,t)\rfloor}{n}\right) \\
&\simeq t
\end{aligned}$$

Thus, $\int_0^{\tau^{-1}(\omega,t)} b\,dW$ has the right variances for a standard Wiener process. It turns out it also has nearly independent increments. It's true that the random variables

$$\frac{b\left(\omega,\frac{k}{n}\right)\omega_{k+1}}{\sqrt{n}}$$

are not independent, but the failure of independence comes from the possible dependence of $b(\omega,k/n)$ on $b(\omega,j/n)\omega_{j+1}$ for $j < k$, and the time change just cancels out this dependence. With independence, the Central Limit Theorem guarantees that the increments are normally distributed.

Now, let's turn back to the continuous-time problem. Let

$$\tau(\omega, t) = \int_0^t b^2(\omega, s)\, ds$$

$\tau(\omega, \cdot)$ is continuous and nondecreasing in time, almost surely; if $b(\omega, t) \neq 0$ almost everywhere, then $\tau(\omega, \cdot)$ is strictly increasing in time, almost surely. Assume

$$\int_0^\infty b^2(\omega, s)\, ds = \infty \text{ almost surely.}$$

Define

$$\tau^{-1}(\omega, t) = \min\{s : \tau(\omega, s) = t\}$$

The minimum exists almost surely, and $\tau\left(\omega, \tau^{-1}(\omega, t)\right) = t$, since $\tau(\omega, \cdot)$ is almost surely continuous in time.[2]

Define

$$\hat{Z}(\omega, t) = Z(\omega, \tau^{-1}(\omega, t))$$

and note that by analogy with the discrete-time calculation, we expect

$$\begin{aligned} \int_0^{\tau^{-1}(\omega,t)} b(\omega, s)^2\, ds &= \tau\left(\tau^{-1}(\omega, t)\right) \\ &= t \end{aligned}$$

and we expect $\hat{Z}$ to normally distributed with mean zero and independent increments, which would make $\hat{Z}$ a standard Wiener process.

Theorem 10.1 (Lévy's Representation Theorem) *Suppose*

$$Z(\omega, t) = \int_0^t b(\omega, s)\, dW$$

[2]If $b(\omega, t) \neq 0$ almost everywhere, $\tau(\omega, \cdot)$ is strictly increasing in t almost surely and $\tau^{-1}\left(\omega, \tau(\omega, t)\right) = t$.

where W is a standard K-dimensional Wiener process and $b \in \mathcal{L}^2$ is $1 \times K$-dimensional. Suppose

$$\tau(\omega, t) = \int_0^t b(\omega, s) b^T(\omega, s)\, ds$$

satisfies

$$\lim_{t \to \infty} \tau(\omega, t) = \infty$$

almost surely. Then

$$\hat{Z}(\omega, t) = Z\left(\omega, \tau^{-1}(\omega, t)\right)$$

is a standard 1-dimensional Wiener process.

3. **Detrending by Adjusting Probabilities (A Girsanov Theorem)** This is a less obvious approach, but it turns out to be very useful in Finance. Let Z be any adapted simple process on the random walk filtration; think of Z as representing the price of a stock, Suppose there is also a money-market account[3] M whose value is given by

$$M\left(\omega, \frac{k}{n}\right) = e^{rk/n}$$

What conditions on Z assure there is no arbitrage? A trader can always borrow (by selling short the bond) and purchase the stock. If s/he borrows $Z(\omega, k/n)$ to buy one unit of the money-market account, s/he will have to pay back $Z(\omega, k/n)e^{r/n}$ at time $\frac{k+1}{n}$. Alternatively, s/he can sell short one share of the stock, place the proceeds $Z(\omega, k/n)$ in the money-market account, and receive $Z(\omega, k/n)e^{r/n}$ at time $\frac{k+1}{n}$. Thus, there is no arbitrage if and only if $Z(\omega, k/n)e^{r/n}$ lies between $Z\left(\omega_-, \frac{k+1}{n}\right)$ and $Z\left(\omega_+, \frac{k+1}{n}\right)$. More precisely, there is no arbitrage if and only if one of the following three mutually exclusive conditions

[3] This is usually referred to as a riskless bond. We follow Nielsen's terminology and call it a money-market account, which is a more accurate description of its properties. Note that M is riskless, whereas a real-world "bond" fluctuates in value, depending on prevailing interest rates.

holds:

$$Z\left(\omega_-, \frac{k+1}{n}\right) < Z\left(\omega, \frac{k}{n}\right) e^{r/n} < Z\left(\omega_+, \frac{k+1}{n}\right)$$
$$Z\left(\omega_-, \frac{k+1}{n}\right) > Z\left(\omega, \frac{k}{n}\right) e^{r/n} > Z\left(\omega_+, \frac{k+1}{n}\right)$$
$$Z\left(\omega_-, \frac{k+1}{n}\right) = Z\left(\omega, \frac{k}{n}\right) e^{r/n} = Z\left(\omega_+, \frac{k+1}{n}\right)$$

Suppose that Z does not admit arbitrage. We shall show how to define the conditional probability $Q_k(\omega_+|\omega)$ to make $Z(\omega, k/n)e^{-rk/n}$ a martingale. There exists a unique[4] $\alpha \in (0, 1)$ such that

$$Z\left(\omega, \frac{k}{n}\right) e^{r/n} = \alpha Z\left(\omega_+, \frac{k+1}{n}\right) + (1 \Leftrightarrow \alpha) Z\left(\omega_-, \frac{k+1}{n}\right)$$

Define $Q_k(\omega_+|\omega)) = \alpha$ and $Q_k(\omega_-|\omega)) = 1 \Leftrightarrow \alpha$; these conditional probabilities define a unique probability measure Q on Ω by

$$Q(\omega) = \prod_{k=0}^{nT-1} Q_k(\omega_{k+1}|\omega_k)$$

Then $Z(\omega, k/n)e^{-rk/n}$ is a martingale with respect to Q. We have proved the following theorem:

Theorem 10.2 *Suppose Z, the stock price, is an adapted simple process with respect to the n-step random walk model, and there is a money-market account*

$$M(\omega, k/n) = e^{rk/n}$$

Suppose that there is no arbitrage. Then there is a probability measure Q such that $Q(\omega) > 0$ for each $\omega \in \Omega$ and such that $Z(\omega, k/n)e^{-rk/n}$ is a martingale with respect to Q.

Now, we turn to the continuous-time case.

[4] If $Z\left(\omega_-, \frac{k+1}{n}\right) = Z\left(\omega, \frac{k+1}{n}\right) = Z\left(\omega_+, \frac{k+1}{n}\right) e^{r/n}$, any $\alpha \in [0, 1]$ will work, so choose arbitrarily $\alpha = 1/2$.

Definition 10.3 We say that a probability measure Q is *absolutely continuous* with respect to P if

$$A \in \mathcal{F}, P(A) = 0 \Rightarrow Q(A) = 0$$

P and Q are said to be *equivalent* if Q is absolutely continuous with respect to P and P is absolutely continuous with respect to Q.

Theorem 10.4 (Radon-Nikodym) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Q is absolutely continuous with respect to P if and only if there is a function $f \in L^1(P)$ (called the Radon-Nikodym derivative of Q with respect to P) such that for all $F \in \mathcal{F}$*

$$Q(F) = \int_F f\, dP$$

Remark 10.5 In Theorem 10.2, it is obvious that $Q(\omega) > 0$ for each $\omega \in \Omega$, so

$$Q(A) = 0 \Leftrightarrow P(A) = 0 \Leftrightarrow Z = \emptyset$$

so P and Q are equivalent. This carries over to the continuous-time case, but equivalence there is considerably more subtle. For example, suppose we found in the random walk case that $Q_k(\omega_-|\omega) = 1/3$ and $Q_k(\omega_+|\omega) = 2/3$ for all k. Then

$$E(X(\omega, 1)) = \frac{n\left(\frac{2}{3} \Leftrightarrow \frac{1}{3}\right)}{\sqrt{n}} = \frac{\sqrt{n}}{3} \to \infty$$

as $n \to \infty$. While the distribution of $X(\cdot, 1)$ is Normal with mean zero and variance 1 under P, the distribution of $X(\cdot, 1)$ blows up under Q, in the limit as $n \to \infty$. Asymptotically, as $n \to \infty$, P and Q become mutually singular; there is are sets $A_n \subset \Omega_n$ such that $P_n(A_n) \to 1$ and $Q_n(A_n) \to 0$.

Now, suppose that Z is an Itô Process

$$Z(\omega, t) = Z(\omega, 0) + \int_0^t a\, ds + \int_0^t b\, dW$$

with $a \in \mathcal{L}^1$ and $b \in \mathcal{L}^2$, $b(\omega, t) \neq 0$. We'll derive a probability measure which makes Z into a martingale. To do this, we pretend that W is an n-step random walk, and a, b are adapted and simple. Then

$$Z\left(\omega, \frac{k+1}{n}\right) = Z\left(\omega, k/n\right) + \frac{a(\omega, k/n)}{n} + \frac{b(\omega, k/n)\omega_{k+1}}{\sqrt{n}}$$

so if Z is to be a martingale,

$$\begin{aligned}
0 &= \frac{a(\omega, k/n)}{n} + \alpha\left(\frac{b(\omega, k/n)}{\sqrt{n}}\right) \Leftrightarrow (1 \Leftrightarrow \alpha)\left(\frac{b(\omega, k/n)}{\sqrt{n}}\right) \\
\frac{a(\omega, k/n)}{n} &= (1 \Leftrightarrow 2\alpha)\left(\frac{b(\omega, k/n)}{\sqrt{n}}\right) \\
\alpha &= \frac{1}{2} \Leftrightarrow \frac{a(\omega, k/n)}{2\sqrt{n}b(\omega, k/n)}
\end{aligned}$$

so

$$\begin{aligned}
Q_k(\omega_+|\omega) &= \frac{1}{2} \Leftrightarrow \frac{a(\omega, k/n)}{2\sqrt{n}b(\omega, k/n)} \\
\frac{Q_k(\omega_+|\omega)}{P(\omega_+|\omega)} &= \frac{\frac{1}{2} \Leftrightarrow \frac{a(\omega, k/n)}{2\sqrt{n}b(\omega, k/n)}}{\frac{1}{2}} \\
&= 1 \Leftrightarrow \frac{a(\omega, k/n)}{\sqrt{n}b(\omega, k/n)} \\
&= 1 \Leftrightarrow \frac{a(\omega, k/n)\omega_{k+1}}{\sqrt{n}b(\omega, k/n)} \\
\frac{Q_k(\omega_-|\omega)}{P(\omega_-|\omega)} &= \frac{\frac{1}{2} + \frac{a(\omega, k/n)}{2\sqrt{n}b(\omega, k/n)}}{\frac{1}{2}} \\
&= 1 + \frac{a(\omega, k/n)}{\sqrt{n}b(\omega, k/n)} \\
&= 1 \Leftrightarrow \frac{a(\omega, k/n)\omega_{k+1}}{\sqrt{n}b(\omega, k/n)}
\end{aligned}$$

Thus,

$$\frac{Q(\omega)}{P(\omega)} = \prod_{k=0}^{nT-1}\left(1 \Leftrightarrow \frac{a(\omega, k/n)\omega_{k+1}}{b(\omega, k/n)\sqrt{n}}\right)$$

$$
\begin{aligned}
\log\left(\frac{Q(\omega)}{P(\omega)}\right) &= \sum_{k=0}^{nT-1} \log\left(1 \Leftrightarrow \frac{a(\omega,k/n)}{b(\omega,k/n)}\frac{\omega_{k+1}}{\sqrt{n}}\right) \\
&= \sum_{k=0}^{nT-1} \left(\Leftrightarrow \frac{a(\omega,k/n)}{b(\omega,k/n)}\frac{\omega_{k+1}}{\sqrt{n}}\right) \Leftrightarrow \frac{1}{2}\left(\Leftrightarrow \frac{a(\omega,k/n)}{b(\omega,k/n)}\frac{\omega_{k+1}}{\sqrt{n}}\right)^2 + O\left(n^{-3/2}\right) \\
&= \Leftrightarrow \frac{1}{2}\int_0^T \frac{a^2(\omega,k/n)}{b^2(\omega,k/n)}\,dt \Leftrightarrow \int_0^T \frac{a(\omega,k/n)}{b(\omega,k/n)}\,dW + O\left(n^{-1/2}\right)
\end{aligned}
$$

This suggests that in the continuous-time problem, the Radon-Nikodym derivative of Q with respect to P should be given by

$$
e^{-\frac{1}{2}\int_0^T \frac{a^2(\omega,t)}{b^2(\omega,t)}\,dt - \int_0^T \frac{a(\omega,t)}{b(\omega,t)}\,dW} \tag{2}
$$

provided that the stochastic integral makes sense (i.e. $\frac{a}{b} \in \mathcal{L}^2$) and the candidate Radon-Nikodym derivative actually defines a *probability* measure on Ω.

Definition 10.6 If $\alpha \in \mathcal{L}^1$ and $\beta \in \mathcal{L}^2$ are 1×1 and $1 \times K$ processes respectively, and W is a K-dimensional standard Wiener process, define

$$
\eta[\alpha,\beta](t) = e^{\int_0^t (\alpha - \frac{1}{2}\beta\beta^T)\,ds + \int_0^t \beta\,dW}
$$

$\eta[\alpha,\beta]$ is called a stochastic exponential

Proposition 10.7 (Proposition 2.18 in Nielsen) *If Z is a 1-dimensional positive Itô process, there exist $\alpha \in \mathcal{L}^1$ and $\beta \in \mathcal{L}^2$ such that*

$$
Z(\omega,t) = Z(\omega,0)\eta[\alpha,\beta]
$$

Proof: Apply Itô's Lemma to $\log X$. ■

Proposition 10.8 (Proposition 2.19 in Nielsen) *If $\alpha \in \mathcal{L}^1$ and $\beta \in \mathcal{L}^2$, then $\eta = \eta[\alpha,\beta]$ is the unique 1-dimensional Itô process satisfying the stochastic differential equation*

$$
\frac{d\eta}{\eta} = \alpha\,dt + \beta\,dW
$$

and the initial condition

$$
\eta(\omega,0) = 1
$$

Suppose we are given an Itô process

$$Z(\omega,t) = Z(\omega,0) + \int_0^t a\,ds + \int_0^t b\,dW$$

We are looking for a probability measure Q so that Z is a martingale with respect to Q. Above, in Equation (2), we conjectured a formula for the Radon-Nikodym derivative of Q with respect to P in the case $N = K = 1$:

$$e^{-\frac{1}{2}\int_0^T \frac{a^2(\omega,t)}{b^2(\omega,t)}\,dt - \int_0^T \frac{a(\omega,t)}{b(\omega,t)}\,dW} = \eta\left[0, \Leftrightarrow\frac{a(\omega,t)}{b(\omega,t)}\right] = \eta[0, \Leftrightarrow\lambda]$$

where[5]

$$\lambda(\omega,t) = \frac{a(\omega,t)}{b(\omega,t)}$$

For general N and K, we replace this condition by $a = b\lambda^T$.

Notice that η is trying very hard to be a martingale; indeed, it will be a martingale provided that

$$e^{-\int_0^T \lambda\,dW} \in \mathcal{H}^2$$

for all T. A sufficient condition for η to be a martingale is the Novikov condition:

$$E\left(e^{\frac{1}{2}\int_0^T \lambda^T\lambda\,ds}\right) < \infty \text{ for all } T$$

We need η to be a martingale to ensure that our formula for the Radon-Nikodym derivative of Q defines a *probability measure*. Indeed, if η is a martingale,

$$\begin{aligned}
Q(\Omega) &= \int_\Omega \eta(T)\,dP \\
&= \int_\Omega E(\eta(T)|\mathcal{F}_0)\,dP \text{ since } \Omega \in \mathcal{F}_0 \\
&= \int_\Omega \eta(0)\,dP \text{ since } \eta \text{ is a martingale} \\
&= \int_\Omega e^0\,dP \\
&= 1
\end{aligned}$$

[5] I have chosen to follow Nielsen's notation here, so please note not to confuse this use of λ with the other use of λ to denote Lebesgue measure.

The converse is also true. If $\int_\Omega \eta[0, \Leftrightarrow\lambda](T)\, dP = 1$ for all T, then $\eta[0, \Leftrightarrow\lambda]$ is a martingale. This follows from the following proposition:

Proposition 10.9 (Proposition 2.23 in Nielsen) *For all $\beta \in \mathcal{L}^2$, $\eta[0, \beta]$ is a supermartingale, i.e. if $s < t$,*

$$E\left(\eta[0, \beta](t) | \mathcal{F}_s\right) \leq \eta[0, \beta](s)$$

Thus, every positive Itô process with zero drift is a supermartingale.

Since η is a supermartingale, if it is not a martingale, we will have $Q(\Omega) < 1$ and Q will not be a *probability* measure. In the discrete setting, the definition automatically made Q be a probability measure, so the assumption that $\eta[0, \Leftrightarrow\lambda]$ is a martingale is saying, in essence, that the discrete process converges to its continuous-time analogue.

Theorem 10.10 (Generalization of Theorem 2.26 in Nielsen) (Revised 4/6/03) *Let W be a K-dimensional standard Wiener process and*

$$Z(\omega, t) = Z(\omega, 0) + \int_0^t a\, ds + \int_0^t b\, dW$$

with $a \in \mathcal{L}^1$ and $b \in \mathcal{L}^2$ of dimension $N \times 1$ and $N \times K$ respectively. Suppose there exists $\lambda \in \mathcal{L}^2$ such that

$$a(\omega, t) = b(\omega, t)\lambda(\omega, t)^T$$

almost everywhere. If $E(\eta[0, \Leftrightarrow\lambda](T)) = 1$[6], then there is a probability measure Q equivalent to P such that each component of Z has zero drift with respect to Q on the interval $[0, T]$. If $b \in \mathcal{H}^2$ with respect to Q, then Z is a vector martingale with respect to Q. The Radon-Nikodym derivative of Q with respect to P is given by

$$\eta[0, \Leftrightarrow\lambda](T)$$

The instantaneous covariance matrix of Z, viewed with respect to Q, is bb^T.

[6] In particular, if the Novikov condition

$$E\left(e^{\frac{1}{2}\int_0^T \lambda^T \lambda\, ds}\right) < \infty$$

is satisfied.

Corollary 10.11 (Theorem 2.26 in Nielsen) *Let W be a K-dimensional Wiener process. Suppose $\lambda \in \mathcal{L}^2$, and define*

$$W^{\lambda}(t) = \int_0^t \lambda^T \, ds + W(t)$$

If $E(\eta[0, \Leftrightarrow\lambda](T)) = 1$, then W^{λ} is a Wiener process with respect to Q, where Q is the probability measure with Radon-Nikodym derivative $\eta[0, \Leftrightarrow\lambda](T)$.

Proof: (Revised 4/6/03) In Theorem 10.10, take $Z(\omega, 0) = 0$, $\lambda^T = a$, and take $b(\omega, t)$ to be the $K \times K$ identity matrix. Then $Z = W^{\lambda}$, $a = b\lambda^T$, $\lambda \in \mathcal{L}^2$, and $E(\eta[0, \Leftrightarrow\lambda](T)) = 1$, so W^{λ} has zero drift under Q by Theorem 10.10; since b is bounded, it is in $\mathcal{H}^2$ with respect to Q, so W^{λ} is a Q-martingale. By Lévy's Representation Theorem (Theorem 10.1), if Z_i is the i^{th} component of Z and τ_i is defined with respect to Z_i as in the statement of Lévy's Representation Theorem, then $Z_i(\omega, \tau_i(\omega, t))$ is a standard 1-dimensional Wiener process under Q, but $\tau_i(\omega, t)$ is t times the (i, i) entry of bb^T, hence $\tau_i(\omega, t) = t$. ∎

11 The Brownian Bridge

In this section, we present an alternative formulation of the Brownian bridge. In the following section, we present an alternative formulation for Ornstein-Uhlenbeck Processes. You should read the rest of Chapter 3 of Nielsen on your own.

Geometric Brownian motion is a reasonably good model of stock pricing, but it is completely inappropriate for some assets. For example, a zero-coupon bond which pays \$1 at time T and no payments at any other time is guaranteed, by arbitrage considerations, to trade for a price of exactly \$1 at time T. On the other hand, fluctuations in interest rates make the value of the bond uncertain at times $t < T$. We want a stochastic process which fluctuates something like a Brownian motion, but is guaranteed to have a particular value at some future time T. The Brownian bridge has exactly these properties; it is also the simplest process based on Brownian motion which exhibits mean reversion. The Brownian bridge is a good model for one long-run zero coupon bond, but unfortunately (as we shall see in a later section) it does not provide a satisfactory foundation for the term structure of interest rates. Ornstein-Uhlenbeck processes, which will be treated in the next section, provide more satisfactory models of interest rates, and mean reversion.

Nielsen defines the Brownian bridge by working through Itô processes. Here, we give an alternative, more hands-on, construction.

Suppose W is a 1-dimensional Wiener process. Fix $\alpha \in \mathbf{R}$ and let $B(t) = W(t|W(T) = \alpha)$. Since $W(T) = \alpha$ is an event of probability zero, one can think of taking

$$B_\varepsilon(t) = W(t|W(T) \in [\alpha \Leftrightarrow \varepsilon, \alpha + \varepsilon])$$

then taking a limit as $\varepsilon \to 0$.

B is a well-defined process, but it is not a Brownian motion. In particular,

- B does *not* have independent increments.
- B is *not* a martingale.
- B is Gaussian; we will see why.

For the sake of exposition, let us assume $\alpha = 0$. The qualitative properties we derive go through for general $\alpha \in \mathbf{R}$. We construct a discrete analogue of the Brownian bridge as follows. Replace W by the n-step binary random walk X, and assume that nT is even, so it is possible to have $X(\omega, t) = 0$. We define

$$B(\omega, t) = X(\omega, t|X(\omega, T) = 0)$$

Clearly, we can think of B is living on

$$\Omega' = \{\omega : X(\omega, T) = 0\} = \left\{\omega : |\{k : \omega_k = +1\}| = \frac{nT}{2}\right\}$$

We can think of Ω' in two other ways:

- It is the set of all permutations of $\frac{nT}{2}$ $+1$'s and $\frac{nT}{2}$ $\Leftrightarrow 1$'s.
- It is the set of all sequences of samples, without replacement, from an urn which initially contains $\frac{nT}{2}$ $+1$'s and $\frac{nT}{2}$ $\Leftrightarrow 1$'s.

This tells us that the Brownian bridge has a symmetry property: If we define $\hat{B}(\omega, t) = B(\omega, T - t)$, then $\hat{B}$ has the same distribution as B. Notice that this means that the process looking forward from time t looks the same as the process looking *backward* from time $T - t$.

We can compute the stochastic differential equation satisfied by B. Suppose that

$$B\left(\omega, \frac{k}{n}\right) = \frac{\ell}{\sqrt{n}}$$

This means that, up to time $\frac{k}{n}$, we have chosen $\frac{k+\ell}{2}$ $+1$'s and $\frac{k-\ell}{2}$ -1's. The remaining draws in the urn consist of $\frac{nT-k-\ell}{2}$ $+1$'s and $\frac{nT-k+\ell}{2}$ -1's. Thus, the probability that ω_{k+1} is $+1$ is

$$\frac{nT - k - \ell}{2(nT - k)}$$

and the probability that ω_{k+1} is -1 is

$$\frac{nT - k + \ell}{2(nT - k)}$$

so the drift per unit time is

$$\begin{aligned}
\frac{\frac{-\ell}{(nT-k)\sqrt{n}}}{\frac{1}{n}} &= \frac{-\ell\sqrt{n}}{nT - k} \\
&= \frac{-\ell\sqrt{n}}{n(T - t)} \\
&= \frac{-\ell}{\sqrt{n}(T - t)} \\
&= \frac{-B(\omega, t)}{(T - t)}
\end{aligned}$$

In other words, the drift at time t points linearly back toward 0 at time T. Similarly, one can calculate the instantaneous variance of the process, which then determines the stochastic differential equation; the details are left to Problem Set 6.

12 Ornstein-Uhlenbeck Processes

Ornstein-Uhlenbeck Processes are very useful for modelling interest rates, as well as other processes which exhibit mean reversion.

Assume that a, b, σ are deterministic constants, $a > 0$, and W is a 1-dimensional Wiener Process. Consider the stochastic differential equation

$$dr = a(b \Leftrightarrow r)\, dt + \sigma\, dW, \quad r(0) = r_0$$

where r_0 is an $\mathcal{F}_0$-measurable random variable or a constant. Notice that if $r > b$, the drift term is negative, while if $r < b$, the drift term is positive. In other words, the process tends to drift back toward b over time, but in addition volatility is constantly being introduced by the $\sigma\, dW$ term. How can we construct a solution and write it as an Itô Process?

Consider the n-step random walk X. Define

$$r\left(\omega, \frac{k}{n}\right) = r_0 + \sum_{j=0}^{k-1} \frac{a\left(b \Leftrightarrow r\left(\omega, \frac{j}{n}\right)\right)}{n} + \sum_{j=0}^{k-1} \frac{\sigma \omega_{j+1}}{\sqrt{n}}$$

r is a perfectly well-defined process in discrete time; if it has a well-defined limit in continuous time, that limit will have to satisfy the stochastic integral equation

$$r(\omega, t) = r_0 + \int_0^t a\left(b \Leftrightarrow r(\omega, s)\right)\, ds + \sigma dW$$

so it will be a solution of the stochastic differential equation, as desired.

In order to see that the discrete solution has a limit, and to get a specific formula for it, consider first the deterministic ordinary differential equation

$$\dot{r} = a(b \Leftrightarrow r), r(0) = r_0$$

The solution is

$$r(t) = b + (r_0 \Leftrightarrow b)e^{-at}$$

as can be seen by computing

$$\begin{aligned} r(0) &= b + (r_0 \Leftrightarrow b) = r_0 \\ \dot{r} &= \Leftrightarrow a(r_0 \Leftrightarrow b)e^{-at} \\ &= \Leftrightarrow a(r \Leftrightarrow b) \\ &= a(b \Leftrightarrow r) \end{aligned}$$

The solutions exhibit a kind of additivity. If $b = 0$, and r and s are solutions of the ordinary differential equation with initial conditions $r(0) = r_0$ and $s(0) = s_0$, then

$$(r + s)(t) = r(t) + s(t) = r_0 e^{-at} + s_0 e^{-at} = (r_0 + s_0)e^{-at}$$

so the solution for the sum of the initial conditions is the sum of the solutions. For general b, we get

$$\begin{aligned} ((r \Leftrightarrow b) + (s \Leftrightarrow b))(t) &= r(t) + s(t) \Leftrightarrow 2b \\ &= b + (r_0 \Leftrightarrow b)e^{-at} + b + (s_0 \Leftrightarrow b)e^{-at} \Leftrightarrow 2b \\ &= ((r_0 \Leftrightarrow b) + (s_0 \Leftrightarrow b))e^{-at} \end{aligned}$$

so when we add the initial conditions (expressed as deviations from b), the solution is the sum of the solutions (again, expressed as deviations from b). This tells us that the initial disturbance from b arising from the initial condition, and the ongoing disturbances from the volatility terms $\omega_j + 1$ enter the solution additively; each disturbance decays exponentially. Thus, we have

$$r\left(\omega, \frac{k}{n}\right) \simeq b + (r_0 \Leftrightarrow b)e^{-ak/n} + \sum_{j=0}^{k-1} \frac{\sigma \omega_{j+1}}{\sqrt{n}} e^{-a(k-j-1)/n}$$

so the solution has a well-defined limit

$$r(\omega, t) = b + (r_0 \Leftrightarrow b)e^{-at} + \sigma \int_0^t e^{-a(t-u)}\, dW(u)$$

It is not hard to check that this is indeed a solution of the stochastic differential equation; see Nielsen for details. Thus, we have the following proposition:

Proposition 12.1 (Proposition 3.8 in Nielsen) *Let a, b, σ be constants, $a > 0$, and r_0 $\mathcal{F}_0$-measurable. The process*

$$\begin{aligned} r(\omega, t) &= b + e^{-at}(r_0 \Leftrightarrow b) + \sigma \int_0^t e^{-a(t-u)}\, dW(u) \\ &= e^{-at} r_0 + \left(1 \Leftrightarrow e^{-at}\right) b + e^{-at} \sigma \int_0^t e^{au}\, dW(u) \end{aligned}$$

is the unique Itô Process satisfying the stochastic differential equation

$$dr = a(b \Leftrightarrow r)\, dt + \sigma\, dW, \quad r(0) = r_0$$

13 Securities and Trading Strategies

We now turn to Chapter 4 of Nielsen. We assume there are $N+1$ long-lived securities indexed $n = 0, \ldots, N$; often but not always, the zeroth security is a Money-Market Account, which is instantaneously riskless. The security price process is an Itô Process

$$\bar{S}(t) = \bar{S}(0) + \int_0^t \bar{\mu}\, ds + \int_0^t \bar{\sigma}\, dW$$

where W is a K-dimensional standard Wiener process, $\bar{\mu} \in \mathcal{L}^1$ is $(N+1) \times 1$, and $\bar{\sigma} \in \mathcal{L}^2$ is $(N+1) \times K$.

A *trading strategy* is an adapted, measurable $1 \times (N+1)$ process $\bar{\Delta}$; $\bar{\Delta}_n(\omega, t)$ denotes the holding of security n at node (ω, t). The *value process* is

$$\bar{\Delta}\bar{S} = \bar{\Delta}_0 \bar{S}_0 + \cdots + \bar{\Delta}_N \bar{S}_N$$

The set of trading strategies for which the capital gain process is well-defined is

$$\mathcal{L}(S) = \{\bar{\Delta} : \bar{\Delta}\bar{\mu} \in \mathcal{L}^1 \text{ and } \bar{\Delta}\bar{\sigma} \in \mathcal{L}^2\}$$

The *Cumulative Gain Process* of $\bar{\Delta}$ with respect to $\bar{S}$ is

$$\begin{aligned}
\mathcal{G}(\bar{\Delta}; \bar{S})(t) &= \bar{\Delta}(0)\bar{S}(0) + \int_0^t \bar{\Delta}\, d\bar{S} \\
&= \bar{\Delta}(0)\bar{S}(0) + \int_0^t \bar{\Delta}\bar{\mu}\, ds + \int_0^t \bar{\Delta}\bar{\sigma}\, dW
\end{aligned}$$

Implicitly, this definition assumes the securities pay no dividends.

$\bar{\Delta}$ is *self-financing* if it satisfies the budget constraint

$$\bar{\Delta}\bar{S} = \mathcal{G}(\bar{\Delta}; \bar{S})$$

i.e.

$$\bar{\Delta}(t)\bar{S}(t) = \bar{\Delta}(0)\bar{S}(0) + \int_0^t \bar{\Delta}\, d\bar{S}$$

almost surely, for all t. In other words, after $\bar{\Delta}(0)\bar{S}(0)$ is invested to buy the initial portfolio $\bar{\Delta}(0)$, no additional money goes in to buy stocks and no money is withdrawn. Writing the self-financing condition in differential form,

$$d(\bar{\Delta}\bar{S}) = \bar{\Delta}\, d\bar{S}$$

i.e. the instantaneous rate of change of the value process is the security holding times the instantaneous rate of change of the security prices. Heuristically, this is saying that

$$\bar{S}\, d\bar{\Delta} = 0$$

i.e. the instantaneous change in the portfolio is orthogonal to the vector of securities prices, which just says the value of shares bought equals the value of shares sold. This is heuristic, rather than precise, because the trading strategy $\bar{\Delta}$ is not required to be an Itô Process, so we may not be able to assign formal meaning to $d\bar{\Delta}$.

It is important to understand that self-financing is a property of the trading strategy and not of the value process. Nielsen gives one example of this. The following is a simpler (but perhaps fairly stupid) example. Suppose there are two assets

$$\bar{S}_0(t) = \bar{S}_0(0)e^{r_0 t} \text{ and } \bar{S}_1(t) = \bar{S}_1(0)e^{r_1 t}$$

with $r_1 > r_0$. The buy-and-hold strategy

$$\bar{\Delta} = (0, 1)$$

(hold one unit of S_2 no matter what) is self-financing and yields the value process

$$\bar{\Delta}(t)\bar{S}(t) = \bar{S}_1(0)e^{r_1 t}$$

The strategy

$$\bar{\Delta}' = \left(\frac{S_1(0)e^{(r_1 - r_0)t}}{S_0(0)}, 0 \right)$$

has the same value process

$$\bar{\Delta}'(t)\bar{S}(t) = \bar{S}_0(0)e^{r_0 t}\frac{\bar{S}_1(0)e^{(r_1 - r_0)t}}{\bar{S}_0(0)} = \bar{S}_1(0)e^{r_1 t}$$

but it is *not* self-financing; money is constantly going in to increase the holding of security zero and this is not balanced by any sale of security one.

The *Cumulative Net Withdrawal Process*[7] is

$$\mathcal{D}(\bar{\Delta}; \bar{S}) = \mathcal{G}(\bar{\Delta}; \bar{S}) \Leftrightarrow \bar{\Delta}\bar{S}$$

[7] The alternative terminology cumulative dividend is very unfortunate; this should refer to the cumulative dividends earned by a portfolio, when the underlying securities pay dividends.

$\bar{\Delta}$ is self-financing if and only if $\mathcal{D}(\bar{\Delta}; \bar{S}) = 0$.

A *numeraire* for $\bar{S}$ is a self-financing trading strategy $\bar{b}$ such that $\bar{b}\bar{S} = 1$ for all t, almost surely. S is said to be *normalized* if there is a numeraire for S. Consider the following examples:

1. Suppose $N = 0$ and $\bar{S}_0$ is a money-market account. The only self-financing trading strategies are buy-and-hold strategies; $\bar{b}(t) = \bar{b}(0)$. If b is a numeraire, then $\bar{S}_0(t)$ must be a constant, independent of t; any increase in value due to interest is incorporated into the units of account in which the security price is measured. Said another way, the currency is not dollars but units of the security, and each unit of the security buys more real goods as time passes.

2. Assuming no new shares are issued or redeemed, and no mergers occur, one self-financing strategy is buy and hold *all* the shares outstanding. If this is a numeraire, then $\bar{\Delta}\bar{S} = 1$ means that prices are deflating at the rate of growth of the market porfolio.

3. At each node (ω, t) we can multiply all the security prices by an arbitrary scalar $\alpha(\omega, t)$ without changing the opportunities to make a self-financed change of portfolio at time t. As long as we also multiply the price of real goods by the same scalar, nothing has changed; the set of goods that can be bought with the value of the portfolio is unchanged. If we multiply security prices at each node by an arbitrary scalar $\alpha(\omega, t)$, the Itô integrals needed to define capital gains may no longer be defined. A surprising fact is that, if the needed stochastic integrals are defined, they compute the capital gains correctly, and the set of self-financing portfolios is invariant to these price changes. We will have more to say on this later.

Example 13.1 [Black-Scholes Model, Example 4.3 in Nielsen] I followed Nielsen closely on this, so I will not reproduce that discussion except to note the following point. The Black-Scholes formula not only tells you how to price a standard call option; it also gives you a necessary condition for any self-financing portfolio that replicates the option. If C is the value of the option and S is the stock price, then the replicating portfolio necessarily holds $\frac{\partial C}{\partial S}$ units of the stock at all times. To see this, note that the uniqueness of coefficients of Itô processes implies that the instantaneous volatility of the replicating portfolio must equal the instantaneous volatility of the option, but since everyting except S in the formula for the option price C varies smoothly over time, the instantaneous volatility of C must be $\frac{\partial C}{\partial S}$ times the instantaneous volatility of S; since the instantaneous volatility of the money-market account is zero, the replicating portfolio must hold exactly $\frac{\partial C}{\partial S}$ units of the stock.

14 Strategies in $\mathcal{L}(\bar{S})$ Admit Arbitrage

In this section, I will give a brief unified treatment of Sections 4.2 and 4.7 of Nielsen. The essential point is this: if we allow traders to hold general strategies in $\mathcal{L}(\bar{S})$, there will be arbitrage opportunities. Hence, we must impose an additional condition ("admissiblity") on trading strategies to rule out arbitrage.

The essential idea is the Harrison-Kreps [3] "doubling strategy," a self-financing strategy which invests zero and yields a payoff greater than or equal to one, almost surely, at time $T = 1$. We will be quite informal. Let us suppose that $\bar{S}$ is an Itô process with constant $\bar{\sigma} \neq 0$, so at least one security has non-zero instantaneous volatility at all times; we will refer to this security as "the stock." Suppose also $\bar{S}_0$ is a money-market account with interest rate $r \geq 0$. The trading strategy is defined inductively as follows:

1. At time $t = 1/2$, set $\bar{\Delta}$ to short the money-market account (i.e. borrow money) and use the proceeds to buy the stock; hold this position until time $t = 3/4$.

2. At time $t = 3/4$, check to see whether $\bar{\Delta}\bar{S} \geq \$1$. If so, close out the portfolio (i.e. sell the stock, use the proceeds to close out the short position on the money-market account, and pocket the profit which is at least \$1). If $\bar{\Delta}\bar{S} < \$1$, short the money-market account further, and use the proceeds to increase the holding of the stock. Hold this new portfolio until time $t = 7/8$.

3. Continue in this way. At time $t = 1 \Leftrightarrow 2^{-n}$, check to see whether $\bar{\Delta}\bar{S} \geq \$1$. If so, close out the portfolio (i.e. sell the stock, use the proceeds to close out the short position on the money-market account, and pocket the profit which is at least \$1.). If $\bar{\Delta}\bar{S} < \$1$, short the

money-market account further, and use the proceeds to increase the holding of the stock. Hold this new portfolio until time $t = 1 \Leftrightarrow 2^{-n-1}$.

Even if the stock's volatility is low, remember that the standard deviation of the stock over a time interval is of the order of the square root of the length of the interval, while any drift in the stock price and any interest owed on the short position in the money-market account over a short interval is proportional to the length of the interval; as the time interval gets short, the volatility overwhelms the interest and drift. Thus, by being sufficiently aggresive at increasing the holding of the stock and increasing the short position in the money-market account, the trader can effectively double her exposure at each time. This transforms the situation into the following situation in roulette:[8] Here is a strategy which wins \$1 for sure at roulette:

1. Borrow \$1, bet \$1 on red. If red comes up, you win \$2; pay off the \$1 you borrowed, pocket \$1, and leave. Otherwise, continue.

2. Borrow another \$2, bet \$2 on red. If red comes up, you win \$4; pay off the total of \$3 that you borrowed, pocket \$1, and leave. Otherwise, continue.

3. At the n^{th} stage, borrow $\$2^{n-1}$ more and bet $\$2^{n-1}$ on red. If red comes up, you win $\$2^n$ dollars, pay off the $\$(1 + 2 + \cdots + 2^{n-1}) = \$(2^n \Leftrightarrow 1)$ total you borrowed, pocket \$1, and leave. Otherwise, continue.

With probability one, you leave with \$1. With probability zero, you never win, and you end up owing an infinite debt. But $0 \times \infty = 0$, so you don't lose sleep over it.

What's wrong with this? There are several ways to see this:

1. At some point, the casino (or your stockbroker) will refuse to extend further credit. Suppose you know from the start the casino will pull the plug after the n^{th} stage. After the n^{th} stage, you will have pocketed \$1 and left with probability $1 \Leftrightarrow 2^{-n}$, but you will owe $\$2^n \Leftrightarrow 1$ with probability 2^{-n}. This is no longer an arbitrage. The refusal to extend further credit is, in effect, a short-sale constraint. Imposing a lower bound on

[8] We assume that the roulette wheel has 36 slots, 18 black and 18 red. In practice, the roulette wheel also has two slots, 0 and 00, which are neither red nor black; the house makes its profit from these.

the trading strategies is perhaps the most realistic way to eliminate arbitrage, but the literature generally shies away from it, because the choice of any particular lower bound seems *ad hoc*, and because short sale constraints make the models less tractable analytically.

2. The Harrison and Kreps doubling strategy is in $\mathcal{L}(\bar{S})$, but not in

$$\mathcal{H}(\bar{S}) = \{b \text{ adapted, measurable} : b\bar{\mu} \in L^1,\, b\bar{\sigma} \in \mathcal{H}^2\}$$

   Indeed, notice that the expected borrowing in the roulette doubling strategy is infinite. As we have noted, stochastic integrals over integrands in $\mathcal{H}^2$ are better behaved than integrals over integrands in $\mathcal{L}^2$; in particular, if $b\bar{\sigma} \in \mathcal{H}^2$, then $\int b\bar{\sigma} dW$ is a martingale. Some papers in the literature have eliminated the doubling strategy by requiring that trading strategies lie in $\mathcal{H}(\bar{S})$. However, assuming that $\bar{\Delta} \in \mathcal{H}(\bar{S})$ is stronger than required, and $\mathcal{H}(\bar{S})$ is not closed under operations we need to do in Finance.

3. If you terminate the roulette doubling strategy at any finite stage, the resulting payoff is a martingale. However, when one goes to the limit and allows infinitely many steps, it is no longer a martingale, essentially because $2^{-n}(2^n \Leftrightarrow 1) = 1 \Leftrightarrow 2^{-n} \to 1$ but by convention $0 \times \infty = 0$. In other words, the infinite doubling strategy is not the natural limit of the finite doubling strategy. One can fix this by restricting attention to a subset of the trading strategies, called the *admissible* trading strategies, whose stochastic integrals *are* martingales. Following Harrison and Kreps [3], this is the approach that has generally been taken in the literature, and it is the one we will use, but it will take us a little while to get there.

15 State Price Processes

In this section, we discuss sections 4.3 and 4.4 of Nielsen.

Definition 15.1 A *state price process* or *pricing kernel* for $\bar{S}$ is a positive 1-dimensional Itô process Π such that $\Pi\bar{S}$ has zero drift.

We should think of a state price process as changing the units in which securities prices are measured at each node; $\Pi\bar{S}$ is the price process in these new units. If Π is a state price process, the $\Pi\bar{S}$ is *almost* but not quite a martingale.

Remark 15.2 Requiring that $\Pi\bar{S}$ have zero drift is mathematically more general than requiring that $\Pi\bar{S}$ be a martingale. It is unclear to me how much this generalization matters to Finance. As we shall see, if Π is a state price process, but $\Pi\bar{S}$ is not a martingale, the strategy of buying and holding the market portfolio will not be admissible, so the set of admissible trading strategies will be constrained in very awkward ways. Perhaps the best reason for generalizing in this way is it is usually easier to verify that $\Pi\bar{S}$ has zero drift than to verify it is a martingale, and making the definition in this way eliminates the need to check that $\Pi\bar{S}$ is a martingale.

In understanding state price processes, it is helpful to relate them to competitive equilibrium. There are severe difficulties in proving existence of equilibrium in continuous-time financial markets; indeed, there are essentially no results with more than one trader in the market. Since there isn't a satisfactory theory of equilibrium, state prices have come to be used as a substitute for equilibrium in the literature. But suppose, as a thought experiment, that the securities prices are the prices in a competitive equilibrium. In other words, we have one or more traders. Each trader has a utility function over consumption in the terminal period T, an endowment process, and initial security holdings. The securities pay a dividend in period T and no dividends in earlier periods. Let us suppose that there is at least one agent whose consumption $c(\omega, T)$ in the terminal period T is interior, i.e. $c(\omega, T) > 0$ for all ω; let u be the utility function of that agent for consumption in period T, and $\bar{\Delta}$ her trading strategy. Then it follows from the first order conditions that

$$\bar{S}(\cdot, t) = \alpha(\cdot, t) E(u'(c(\cdot, T))\bar{S}(\cdot, T)|\mathcal{F}_t) \tag{3}$$

To see this, note that if

$$\frac{\bar{S}_n(\omega, t)}{E(u'(c(\cdot, T))\bar{S}_n(\cdot, T)|\mathcal{F}_t)(\omega)} < \frac{\bar{S}_m(\omega, t)}{E(u'(c(\cdot, T))\bar{S}_m(\cdot, T)|\mathcal{F}_t)(\omega)}$$

then the given agent can achieve a higher utility level by modifying $\bar{\Delta}$ by buying a little more of $\bar{S}_n$ (or shorting it a little less) at the node (ω, t) and buying a little less of $\bar{S}_m$ (or shorting it more) and holding this position, in addition to the holding prescribed by $\bar{\Delta}$, in all nodes that follow (ω, t). Consumption at times $t < T$ is not changed. When the portfolio is liquidated at T, the consumption at period T yields a higher expected utility than the consumption obtained by following $\bar{\Delta}$, so $\bar{\Delta}$ could not have been the agent's equilibrium trading strategy. The factor $\alpha(\omega, t)$ is needed because equilibrium can only set the relative prices of securities at the node (ω, t); one can normalize by multiplying all the prices at a given node by an arbitrary constant, without changing the self-financing trading strategies or the budget set of the agent.

Let

$$\Pi(\omega, t) = \frac{1}{\alpha(\omega, t)}$$

Then

$$\Pi(\omega, t)\bar{S}(\omega, t) = E(u'(c(\cdot, T))\bar{S}(\cdot, T)|\mathcal{F}_t)(\omega)$$

and this *is* a martingale (and thus Π is a state price process) provided that

$$E(|u'(c(\cdot, T))\bar{S}(\cdot, T)|) < \infty$$

When the price process $\bar{S}$ is presented in the form of Equation (3), risk-adjustment is already represented in $\bar{S}$ by the factor $u'(c(\omega, T))$. Π incorporates time-discounting. One could use the factor $u'(c(\omega, T))$ to derive the risk-adjusted probabilities, alternatively incorporate the risk-adjustment into Π.

Thus, if $\bar{S}$ were an *equilibrium* price process, then (modulo the issues hinted at in the above discussion) there would be a state price Π. State prices have proven very useful in the literature as a finesse around the problem that we don't have a remotely satisfactory theory of existence of equilibrium in continuous-time finance.

Now, we return to the material in Nielsen. Recall we are assuming that the security price process $\bar{S}$ satisfies the stochastic differential equation

$$d\bar{S} = \bar{\mu}\, dt + \bar{\sigma}\, dW$$

Since Π is a positive Itô process

$$\Pi = \Pi(0)\eta[\Leftrightarrow r, \Leftrightarrow \lambda]$$

for some $\Pi(0) > 0$, $r \in \mathcal{L}^1$, and $\lambda \in \mathcal{L}^2$, i.e.

$$\begin{aligned}
\Pi(t) &= \Pi(0)e^{\int_0^t (-r-\lambda\lambda^T/2)\,ds - \int_0^t \lambda\,dW} \\
\frac{d\Pi}{\Pi} &= \Leftrightarrow r\,dt \Leftrightarrow \lambda\,dW
\end{aligned}$$

r represents riskless time discounting; it is called the *instantaneously riskless interest rate*, λ is called the $(1 \times K)$ vector of *prices of risk*. Notice that there is a price of risk for each component of the underlying Wiener process W, not a price of risk for each stock. As we shall see, the risk premium of each stock can be derived from its coefficients on the underlying Wiener process W and the prices of risk of the components of W. Indeed, if the risk premium cannot be derived in this way, there will be an arbitrage.

Proposition 15.3 *If* Π *is a state price process for* $\bar{S}$ *and* $\bar{\Delta} \in \mathcal{L}(\bar{S})$ *is self-financing, then* $\Pi\bar{\Delta}\bar{S}$ *has no drift.*

Proof: This is mostly an exercise in Itô's Lemma for Products and the self-financing constraint.

$$\begin{aligned}
d(\Pi\bar{\Delta}\bar{S}) &= \Pi d(\bar{\Delta}\bar{S}) + \bar{\Delta}\bar{S}\,d\Pi + (d\Pi)d(\bar{\Delta}\bar{S}) \\
&= \Pi\bar{\Delta}\,d\bar{S} + \bar{\Delta}\bar{S}\,d\Pi + \Pi(\Leftrightarrow r\,dt \Leftrightarrow \lambda\,dW)(\bar{\Delta}\,d\bar{S}) \\
&\quad \text{(using the self-financing constraint twice,} \\
&\quad \ \text{and the stochastic differential equation satisfied by } \Pi) \\
&= \Pi\bar{\Delta}\,d\bar{S} + \bar{\Delta}\bar{S}\,d\Pi + \Pi\bar{\Delta}(\Leftrightarrow r\,dt \Leftrightarrow \lambda\,dW)(\bar{\mu}\,dt + \bar{\sigma}\,dW) \\
&= \Pi\bar{\Delta}\,d\bar{S} + \bar{\Delta}\bar{S}\,d\Pi \Leftrightarrow \Pi\bar{\Delta}\bar{\sigma}\lambda^T\,dt \\
&\quad \text{(since } (dW)^2 = dt \text{ and } (dt)^2 = dW\,dt = 0) \\
&= \bar{\Delta}(\Pi\,d\bar{S} + \bar{S}\,d\Pi + (d\Pi)(d\bar{S})) \\
&= \bar{\Delta}\,d(\Pi\bar{S})
\end{aligned}$$

Since $\Pi\bar{S}$ has no drift, $\Pi\bar{\Delta}\bar{S}$ has no drift.■

Suppose now that there is a money-market account embedded in $\bar{S}$. This means either

1. the zeroth security S_0 is instantaneously riskless, i.e.

$$\frac{d\bar{S}_0(\omega,t)}{\bar{S}_0(\omega,t)} = r(\omega,t)\,dt$$

or

2. there is a self-financing trading strategy $\bar{b}$ whose value process $M = \bar{b}\bar{S}$ is instantaneously riskless.

For example, if the basic securities explicitly listed as components of $\bar{S}$ include a stock and a call option on the stock *priced by the Black-Scholes formula*, then we can construct a money market account.[9]

If $\bar{b}$ is a money-market account, then $\bar{b}\bar{\sigma} = 0$ for almost all (ω, t), so the $(N+1) \times K$ matrix $\bar{\sigma}$ must have rank at most N almost everywhere. Notice that if $K \leq N$, this necessary condition for the existence of a money-market account follows automatically. This raises a modeling question: Which is bigger, K or N. Normally, we assume that the number of securities is small compared to the number of sources of uncertainty, particularly if one considers individualized uncertainty. On the other hand, Arbitrage Pricing Theory tries to identify a comparatively small number of "factors" $W_1, \ldots, W_K$ that explain most of the variation in securities pricing. For example, if there are N securities, and

$$S_n = e^{(\bar{\mu}_n - \sum_{k=1}^{\infty} \sigma_{nk}^2/2)t + \sum_{k=1}^{\infty} \bar{\sigma}_{nk} d\bar{W}_k}$$

with

$$\sum_{k=1}^{\infty} \sigma_{nk}^2 < \infty$$

most of the variation in S_n can be explained by a finite number of the components of $\bar{W}$; this suggests one might be able to construct an approximate money-market account in some sense.

If $M = \bar{b}\bar{S}$ is a money-market account, then $\bar{b}$ is self-financing and

$$M = M(0)\eta[\alpha, 0]$$

for some $M(0) > 0$ and $\alpha \in \mathcal{L}^1$. Therefore,

$$\begin{aligned} \Pi M &= \Pi(0)M(0)\eta[\Leftrightarrow r, \Leftrightarrow \lambda]\eta[\alpha, 0] \\ &= \Pi(0)M(0)\eta[\alpha \Leftrightarrow r, \Leftrightarrow \lambda] \end{aligned}$$

[9]This argument begs the question of why, if there is no money-market account, the option should be priced by the Black-Scholes formula. The price of the option is forced by arbitrage to be the Black-Scholes formula if there is a money-market account, but in the absence of a money-market account, it presumably could be priced in a different way without creating arbitrage opportunities.

Since $M = \bar{b}\bar{S}$, $\bar{b}$ is self-financing, and Π is a state-price process, the drift of ΠM is zero, i.e.

$$\begin{aligned}
\alpha \Leftrightarrow r &= 0, \;\; \alpha = r \\
M(t) &= M(0)\eta[r, 0](t) \\
&= M(0)e^{\int_0^t r\, ds} \\
\frac{dM}{M} &= r\, dt
\end{aligned}$$

In other words, if there is a money-market account M, the drift of the state-price process must equal the interest rate on M. r is the *instantaneous rate of return* on the money-market acount; note it is a function of (ω, t), not a constant. We write

$$M = M(0)\eta[\Leftrightarrow r, 0]$$

even if M is not available in the market under the prices $\bar{S}$, i.e. even if there does not exist $\bar{b}$ self-financing such that $\bar{b}\bar{S} = M$; note that r comes from the state price process Π. By Itô's Lemma, since $\Pi = \Pi(0)\eta[\Leftrightarrow r, \Leftrightarrow\lambda]$, so $d\Pi = \Leftrightarrow \Pi r\, dt \Leftrightarrow \Pi\lambda\, dW$, we have

$$\begin{aligned}
d(\Pi\bar{S}) &= \Pi\, d\bar{S} + \bar{S}\, d\Pi + (d\Pi)(d\bar{S}) \\
&= \Pi(\bar{\mu}\, dt + \bar{\sigma}\, dW) + \bar{S}\Pi(\Leftrightarrow r\, dt \Leftrightarrow \lambda\, dW) \Leftrightarrow \Pi\sigma\lambda^T\, dt
\end{aligned}$$

the drift of $\Pi\bar{S}$ is

$$\Pi[\bar{\mu} \Leftrightarrow r\bar{S} \Leftrightarrow \bar{\sigma}\lambda^T]$$

Thus, Π is a state-price process for S if and only if

$$\bar{\mu} \Leftrightarrow r\bar{S} = \bar{\sigma}\lambda^T \tag{4}$$

if and only if

$$(\bar{\mu}_n \Leftrightarrow r)\bar{S}_n = \sum_{k=1}^{K} \bar{\sigma}_{nk}\lambda_k$$

In other words, the excess return on security n is the inner product of the vector of coefficients of S_n on W and the vector of prices of risk of the components of W. This strong and useful condition is in essence an arbitrage condition. Note that

$$d\bar{S}_n = \bar{\mu}_n\, dt + \sum_{k=1}^{K} \bar{\sigma}_{nk}\, dW_k$$

If we had m and n such that $\bar{\sigma}_{nk} = \bar{\sigma}_{mk}$ for each k, then $\bar{S}_m$ and $\bar{S}_n$ would have identical volatility; if $\bar{\mu}_m > \bar{\mu}_n$, we can obtain an arbitrage by shorting $\bar{S}_n$ and going long in $\bar{S}_n$. This analysis can be extended to create an arbitrage if Equation (4) is not satisfied.

The instantaneous variance of Π is $\lambda\lambda^T$, so high prices of risk imply that Π is very volatile.[10]

If $\bar{\Delta}$ is a self-financing trading strategy (in particular, buy and hold one security), the instantaneous excess return is

$$\bar{\Delta}\bar{\mu} \Leftrightarrow r\bar{\Delta}\bar{S} = \bar{\Delta}(\bar{\mu} \Leftrightarrow r\bar{S}) = \bar{\Delta}\bar{\sigma}\lambda^T$$

But

$$\frac{d\Pi}{\Pi} = \Leftrightarrow r\,dt \Leftrightarrow \lambda\,dW$$

so portfolios that are negatively correlated with changes in Π have higher expected returns. Notice that this is connected in spirit with one of the main conclusions of the Capital Asset Pricing Model; the excess return on a security is not determined by its volatility, but rather by its covariance with the market portfolio, which determines the portion of its volatility which cannot be diversified away.

Recall that

$$\Pi M = \Pi(0)M(0)\eta[0, \Leftrightarrow\lambda]$$

so

$$\Pi = \frac{\Pi(0)M(0)\eta[0, \Leftrightarrow\lambda]}{M}$$

and

$$\frac{\Pi(T)}{\Pi(t)} = \frac{\eta[0, \Leftrightarrow\lambda](T)/\eta[0, \Leftrightarrow\lambda](t)}{M(T)/M(t)}$$

Prices adjust for the time value of money and for risk; $\eta[0, \Leftrightarrow\lambda]$ is the risk adjustment. We will see in Chapter 5 that the "risk-adjusted probabilities" are formed by incorporating the risk adjustment $\eta[0, \Leftrightarrow\lambda]$ into the probability measure, making $\frac{\bar{S}(t)}{M(t)}$ into a martingale.

If $\bar{S}$ is a normalized securities price process (recall this means that there is a self-financing trading strategy $\bar{b}$ such that $\bar{b}\bar{S} = 1$, so $\bar{b}$ is a money-market

[10] In an equilibrium context, high prices of risk mean that agents are risk-averse, and the risk is undiversifiable.

account with $r = 0 = \alpha$), Π is a state price process for $\bar{S}$ if and only if $\Pi = \Pi(0)\eta[0, \Leftrightarrow\lambda]$ where λ is a system of prices of risk for $\bar{S}$, i.e. $\lambda \in \mathcal{L}^2$, $\bar{\mu} = \bar{\sigma}\lambda^T$.

16 Existence and Uniqueness of State Price Processes

The existence and uniqueness of a state price process is equivalent to the existence and uniqueness of $r \in \mathcal{L}^1$ and $\lambda \in \mathcal{L}^2$ such that

$$\bar{\mu} \Leftrightarrow r\bar{S} = \bar{\sigma}\lambda^T$$

If there is a money-market account, r must be the interest rate on the account.

If $K > N + 1$, there is no hope of uniqueness; $N + 1$ security prices do not contain enough information to price the risk of the K components of the Wiener process W uniquely.

Example 16.1 Suppose $N = 1$. Security zero is a money-market account M, and security one is a stock S whose stochastic differential equation is

$$dS = \mu(W_2)\,dt + \sigma(W_2)\,dW_1$$

In other words, the stock price is driven by W_1, the first component of the Wiener process, but the instantaneous mean and volatility are functions of W_2, which is independent from W_1. In particular, the volatility of the stock price is stochastic. This is important because stochastic volatility permits a better fit to empirically observed stock price processes, which exhibit periods of relatively high volatility and periods of relatively low volatility. Stochastic volatility is also a means of fattening the tails of the stock return distribution compared to those obtained from a log normal with constant volatility. However, if volatility is stochastic in this manner, markets are *not* dynamically complete, and options cannot be uniquely priced by arbitrage considerations alone.

In order for a state price process to exist, it must be possible at each (ω, t) to find a solution to $\bar{\mu} \Leftrightarrow r\bar{S} = \bar{\sigma}\lambda^T$. Notice that

$$\exists_\lambda\ \bar{\mu} \Leftrightarrow r\bar{S} = \bar{\sigma}\lambda^T \Leftrightarrow \bar{\mu} \Leftrightarrow r\bar{S} \in \text{span}\ \{\bar{\sigma}_1, \ldots, \bar{\sigma}_K\}$$

where $\bar{\sigma}_k$ denotes the k^{th} column of $\bar{\sigma}$; in this case, λ^T is just the vector of coefficients expressing $\bar{\mu} \Leftrightarrow r\bar{S}$ as a linear combination of the columns of $\bar{\sigma}$.[11] In addition, the solutions to these linear equations at each (ω, t) must fit together into processes in $\mathcal{L}^1$ and $\mathcal{L}^2$.

Proposition 16.2 (Proposition 4.4 in Nielsen, No Money-Market Account) *Assume $K \geq N+1$. Suppose* rank $\bar{\sigma} = N+1$ *almost everywhere. Let $r \in \mathcal{L}^1$. Set*

$$\lambda^* = (\bar{\mu} \Leftrightarrow r\bar{S})^T (\bar{\sigma}\bar{\sigma}^T)^{-1} \bar{\sigma}$$

There exists a vector of prices of risk (for r) if and only if $\lambda^* \in \mathcal{L}^2$. In that case,

$$|\lambda^*(\omega, t)| \leq |\lambda(\omega, t)| \text{ at almost all } (\omega, t)$$

for every vector of prices of risk λ corresponding to r. If $K = N+1$ and $\lambda^* \in \mathcal{L}^2$, then λ^* is the unique vector of prices of risk corresponding to r.

Proposition 16.3 (Proposition 4.5 in Nielsen, Money-Market Account) *Assume $K \geq N$, the zeroth security is a money-market account with interest rate $r \in \mathcal{L}^1$, and* rank $\bar{\sigma} = N$. *Let*

$$\tilde{\sigma} = \begin{pmatrix} \bar{\sigma}_{11} & \cdots & \bar{\sigma}_{1K} \\ \vdots & & \vdots \\ \bar{\sigma}_{N1} & \cdots & \bar{\sigma}_{NK} \end{pmatrix} \quad \tilde{\mu} = \begin{pmatrix} \bar{\mu}_1 \\ \vdots \\ \bar{\mu}_N \end{pmatrix} \quad \tilde{S} = \begin{pmatrix} \bar{S}_1 \\ \vdots \\ \bar{S}_N \end{pmatrix}$$

Set

$$\lambda^* = (\tilde{\mu} \Leftrightarrow r\tilde{S})^T (\tilde{\sigma}\tilde{\sigma}^T)^{-1} \tilde{\sigma}$$

There exists a vector of prices of risk (for r) if and only if $\lambda^* \in \mathcal{L}^2$. In that case,

$$|\lambda^*(\omega, t)| \leq |\lambda(\omega, t)| \text{ at almost all } (\omega, t)$$

for every vector of prices of risk λ corresponding to r.

Remark 16.4 λ^* is called the minimal vector of state prices, because it is of minimal length for each (ω, t); $\eta[\Leftrightarrow r, \Leftrightarrow \lambda^*]$ is called the minimal state price process because it has minimal volatility. The minimal state price process is sometimes identified in the literature as a focal state price process, and used to pick a focal valuation of derivatives. In my view, this is completely unjustified, unless the state price process is actually unique.

[11] Nielsen expresses this condition as the rank of $\bar{\sigma}$ equals the rank of a matrix with $\bar{\mu} - r\bar{S}$ in the first column and the columns of $\bar{\sigma}$ in the second through $K+1^{st}$ columns.

Proof: (of Proposition 4.4) Since rank $\bar{\sigma} = N + 1$, there is a solution to the equation

$$\bar{\mu} \Leftrightarrow r\bar{S} = \bar{\sigma}\lambda^T \tag{5}$$

We will verify that the formula given for λ^* is a solution. Since rank $\bar{\sigma} = N + 1$,

$$x\bar{\sigma} = 0 \Rightarrow x = 0$$

Thus, if x is a nonzero $(N+1)$-dimensional row vector,

$$\begin{aligned} x(\bar{\sigma}\bar{\sigma}^T)x^T &= (x\bar{\sigma})(x\bar{\sigma})^T = |x\bar{\sigma}|^2 > 0 \\ &\Rightarrow (\bar{\sigma}\bar{\sigma}^T)x^T \neq 0 \end{aligned}$$

which implies that $\bar{\sigma}\bar{\sigma}^T$ is nonsingular, so the formula

$$\lambda^* = (\bar{\mu} \Leftrightarrow r\bar{S})^T(\bar{\sigma}\bar{\sigma}^T)^{-1}\bar{\sigma}$$

makes sense and

$$\begin{aligned} \bar{\sigma}(\lambda^*)^T &= \left(\lambda^*\bar{\sigma}^T\right)^T \\ &= \left((\bar{\mu} \Leftrightarrow r\bar{S})^T(\bar{\sigma}\bar{\sigma}^T)^{-1}\bar{\sigma}\bar{\sigma}^T\right)^T \\ &= \left((\bar{\mu} \Leftrightarrow r\bar{S})^T\right)^T \\ &= \bar{\mu} \Leftrightarrow r\bar{S} \end{aligned}$$

Let V be the span of the rows $\bar{\sigma}_0, \ldots, \bar{\sigma}_N$ of $\bar{\sigma}$ and X the orthogonal complement

$$\begin{aligned} X &= \{x \in \mathbf{R}^K : x \cdot v = 0 \text{ for all } v \in V\} \\ &= \left\{x \in \mathbf{R}^K : x \cdot \bar{\sigma}_n = 0,\ 0 \leq n \leq N\right\} \end{aligned}$$

Given two solutions λ, λ' to the linear equation (5),

$$\bar{\sigma}(\lambda \Leftrightarrow \lambda')^T = 0$$

so $\lambda \Leftrightarrow \lambda'$ is perpendicular to the rows of $\bar{\sigma}$, i.e. $\lambda \Leftrightarrow \lambda' \in X$. On the other hand, if λ is a solution to the linear equation (5), $\lambda + x$ is also a solution, for every $x \in X$. Thus, the set of solutions of the linear equation is exactly

$$\lambda^* + X$$

If $K = N + 1$, $X = \{0\}$ and λ^* is the unique solution of the linear equation (5).

Notice that
$$\lambda^* = \left[(\bar{\mu} \Leftrightarrow r\bar{S})^T(\bar{\sigma}\bar{\sigma}^T)^{-1}\right]\bar{\sigma}$$
is a linear combination of the rows of $\bar{\sigma}$, hence $\lambda^* \in V$, so λ is perpendicular to X. Thus, if $x \in X$,

$$|\lambda^* + x|^2 = |\lambda^*|^2 + |x|^2 \geq |\lambda^*|^2$$

so λ^* is the minimal solution of the linear equation (5).

Now, suppose $\lambda \in \mathcal{L}^2$ is a process which solves the linear equation (5) almost everywhere. In particular, λ is an adapted measurable process, and $\lambda^*(\omega, t)$ is the orthogonal projection of it onto $V_{\omega,t}$, the space spanned by the rows of $\bar{\sigma}(\omega, t)$. Clearly, this makes λ^* adapted. Nielsen tacitly assumes this makes λ^* measurable, and I will duck this question also (but I have no doubt that it does). Since $|\lambda^*(\omega, t)|^2 \leq |\lambda(\omega, t)|^2$ almost everywhere, this shows $\lambda^* \in \mathcal{L}^2$. ■

Example 16.5 [Example 4.7 in Nielsen] This is the Ball-Torous Model, with two zero-coupon bonds, with maturity dates $T_1 < T_2$. The bond prices follow Brownian bridge processes. The model seeks to price an option on the long bond which expires at the maturity date T_1 of the short bond. Nielsen shows there is no state price process in this model; the esential point is that the long and short interest rates need to be linked in a way that is impossible for two (correlated) Brownian motions; the problem occurs at time T_1.

17 Arbitrage and Admissibility

We saw that the value process of the doubling strategy is not a martingale. This leads us to the following definition.

Definition 17.1 A *self-financing arbitrage strategy* is a self-financing trading strategy $\bar{\Delta} \in \mathcal{L}(\bar{S})$ such that either

1. $\bar{\Delta}(0)\bar{S}(0) < 0$ almost surely and for some t, $\bar{\Delta}(t)\bar{S}(t) \geq 0$ almost surely; or

2. $\bar{\Delta}(0)\bar{S}(0) \leq 0$ almost surely and for some t, $\bar{\delta}(t)\bar{S}(t) \geq 0$ almost surely and $\bar{\Delta}(t)\bar{S}(t) > 0$ with positive probability.

Definition 17.2 A self-financing trading strategy $\bar{\Delta} \in \mathcal{L}(\bar{S})$ is *admissible* (for $\bar{S}$ and state price process Π) if $\Pi\bar{\Delta}\bar{S}$ is a martingale.

Remark 17.3 Notice that admissibility is a property of $\bar{\Delta}$ *and* of Π. The existence of an admissible strategy need not imply that $\Pi\bar{S}$ is a martingale; that's an issue of whether buy-and-hold strategies are admissible, and the theory works without assuming that. Note, however, that a finance model in which buy-and-hold strategies are *not* self-financing is a weird model; it is not clear that such models are interesting.

Proposition 17.4 *A self-financing arbitrage strategy is not admissible.*

Proof: If $\bar{\Delta}$ a self-financing arbitrage strategy, there exists t such that

$$E(\Pi(t)\bar{\Delta}(t)\bar{S}(t) > \Pi(0)\bar{\Delta}(0)\bar{S}(0)$$

so $\Pi\bar{\Delta}\bar{S}$ cannot be a martingale. ■

Proposition 17.5 (Proposition 4.11 in Nielsen) *Let Π be a state price process for $\bar{S}$. There is no self-financing arbitrage strategy $\bar{\Delta}$ such that $\Pi\bar{\Delta}\bar{S}$ is bounded below.*

Proof: Suppose $\bar{\Delta}$ is a self-financing arbitrage strategy such that $\Pi\bar{\Delta}\bar{S}$ is bounded below. Choose K such that

$$\Pi(t)\bar{\Delta}(t)\bar{S}(t) \geq K \text{ for all } t$$

almost surely. Then $\Pi\bar{\Delta}\bar{S} \Leftrightarrow K$ is a positive Itô process with zero drift, hence it is a supermartingale by Proposition 2.23 of Nielsen. Since $\bar{\Delta}$ is a self-financing arbitrage strategy, there exists t such that

$$E(\Pi(t)\bar{\Delta}(t)\bar{S}(t)) > E(\Pi(0)\bar{\Delta}(0)\bar{S}(0))$$

so $\Pi\bar{\Delta}\bar{S}$ is not a supermartingale, contradiction. ■

18 Changing the Units of Account

We already touched on this when we talked about a numeraire, and when we saw that if the pricing process arose from an interior equilibrium, we could find a state price process. This section focuses on using a positive Itô process as the numeraire.

Suppose that $E(t) = E(0)\eta[\mu_E, \sigma_E](t)$, where $\mu_E \in \mathcal{L}^1$ is 1-dimensional, $\sigma_E \in \mathcal{L}^2$ is $1 \times K$-dimensional, and

$$\frac{dE}{E} = \mu_E \, dt + \sigma_E \, dW$$

E is the price of the new unit of acount in terms of the old unit of account.

Example 18.1 Suppose that E is the price of a foreign currency in terms of the home currency. Then $\bar{S}$ is the price of the securities in the home currency and $\bar{S}/E$ is the price of the securities in the foreign currency.

By the Itô Quotient Rule (Example 2.15 of Nielsen) ,

$$\begin{aligned} d(\bar{S}/E) &= \frac{1}{E}\left[\bar{\mu} + \bar{S}(\Leftrightarrow \mu_E + \sigma_E\sigma_E^T) \Leftrightarrow \bar{\sigma}\sigma_E^T\right] dt + \frac{1}{E}[\bar{\sigma} \Leftrightarrow \bar{S}\sigma_E] \, dW \\ &= \frac{1}{E}\left[\bar{\mu} \Leftrightarrow \bar{S}\mu_E \Leftrightarrow (\bar{\sigma} \Leftrightarrow \bar{S}\sigma_E)\sigma_E^T\right] dt + \frac{1}{E}[\bar{\sigma} \Leftrightarrow \bar{S}\sigma_E] \, dW \end{aligned}$$

Proposition 18.2 (Proposition 4.12 in Nielsen) *Let E be a positive Itô process. A trading strategy $\bar{\Delta}$ is self-financing with respect to $\bar{S}$ if and only if it is self-financing with respect to $\bar{S}/E$.*

Remark 18.3 Indeed, given an *arbitrary* positive process E, the self-financing constraint is not changed at any given (ω, t) by change the prices from $\bar{S}$ to $\bar{S}/E$. In particular, in discrete time, since the self-financing constraint is applied one node at a time, this proposition is true for an arbitrary positive process E. In continuous time, we need assumptions to ensure that we can compute the stochastic integrals in the self-financing constraint.

Proof: Suppose $\bar{\Delta}$ is self-financing with respect to $\bar{S}$. We first check that $\bar{\Delta} \in \mathcal{L}(\bar{S}/E)$. As we saw,

$$d(\bar{S}/E) = \frac{1}{E}\left[\bar{\mu} \Leftrightarrow \bar{S}\mu_E \Leftrightarrow (\bar{\sigma} \Leftrightarrow \bar{S}\sigma_E)\sigma_E^T\right] dt + \frac{1}{E}[\bar{\sigma} \Leftrightarrow \bar{S}\sigma_E] \, dW$$

so we need to show

$$\frac{\bar{\Delta}}{E}\left[\bar{\mu} \Leftrightarrow \bar{S}\mu_E \Leftrightarrow (\bar{\sigma} \Leftrightarrow \bar{S}\sigma_E)\sigma_E^T\right] = \frac{1}{E}\left[\bar{\Delta}\bar{\mu} \Leftrightarrow \bar{\Delta}\bar{S}\mu_E \Leftrightarrow (\bar{\Delta}\bar{\sigma} \Leftrightarrow \bar{\Delta}\bar{S}\sigma_E)\sigma_E^T\right] \in \mathcal{L}^1 \tag{6}$$

and

$$\frac{\bar{\Delta}}{E}\left[\bar{\sigma} \Leftrightarrow \bar{S}\sigma_E\right] = \frac{1}{E}\left[\bar{\Delta}\bar{\sigma} \Leftrightarrow \bar{\Delta}\bar{S}\sigma_E\right] \in \mathcal{L}^2 \tag{7}$$

For Equation (6), note that

- E is continuous and positive, hence almost surely uniformly bounded away from zero on finite time intervals $[0, T]$, so $1/E$ is almost surely bounded on finite time intervals
- $\bar{\Delta}\bar{\mu} \in \mathcal{L}^1$ since $\bar{\Delta} \in \mathcal{L}(S)$
- $\bar{\Delta}\bar{S}$ is continuous because $\bar{\Delta}$ is self-financing, hence $\bar{\Delta}\bar{S}$ is an Itô Integral; therefore, $\bar{\Delta}\bar{S}$ is almost surely uniformly bounded on finite time intervals $[0, T]$
- $\mu_E \in \mathcal{L}^1$, by the definition of μ_E, so $\bar{\Delta}\bar{S}\mu_E \in \mathcal{L}^1$ because $\bar{\Delta}\bar{S}$ is almost surely uniformly bounded on finite time intervals
- $\bar{\Delta}\bar{\sigma} \in \mathcal{L}^2$ because $\bar{\Delta}$ is a trading strategy; $\sigma_E \in \mathcal{L}^2$ by the definition of σ_E, so

  $$\bar{\Delta}\bar{\sigma}\sigma_E^T \in \mathcal{L}^1 \text{ and } \bar{\Delta}\bar{S}\sigma_E\sigma_E^T \in \mathcal{L}^1$$

  since $\bar{\Delta}\bar{S}$ is almost surely bounded on finite time intervals and the produce of elements of $\mathcal{L}^2$ lies in $\mathcal{L}^1$.

so Equation (6) is satisfied.

For Equation (7), note that

- as before, $1/E$ is almost surely uniformly bounded on finite time intervals $[0, T]$
- $\bar{\Delta}\bar{\sigma} \in \mathcal{L}^2$ since $\bar{\Delta}$ is a trading strategy
- as before, $\bar{\Delta}\bar{S}$ is almost surely uniformly bounded on finite time intervals $[0, T]$
- $\sigma_E \in \mathcal{L}^2$, by the definition of σ_E, so $\bar{\Delta}\bar{S}\sigma_E \in \mathcal{L}^2$

so Equation (7) is satisfied and $\bar{\Delta} \in \mathcal{L}(\bar{S}/E)$.

Now, we show that $\bar{\Delta}$ is self-financing with respect to $\bar{S}/E$.

$$\begin{aligned} d\left(\frac{\bar{\Delta}\bar{S}}{E}\right) & \\ &= \frac{1}{E}d(\bar{\Delta}\bar{S}) + \bar{\Delta}\bar{S}d\left(\frac{1}{E}\right) \Leftrightarrow \frac{1}{E}\bar{\Delta}\bar{\sigma}\sigma_E^T\,dt \\ &\quad \text{by Itô's Lemma} \\ &= \frac{1}{E}\bar{\Delta}d\bar{S} + \bar{\Delta}\bar{S}d\left(\frac{1}{E}\right) \Leftrightarrow \frac{1}{E}\bar{\Delta}\bar{\sigma}\sigma_E^T\,dt \\ &\quad \text{since } \bar{\Delta} \text{ is self-financing} \\ &= \bar{\Delta}\left[\frac{1}{E}\,d\bar{S} + \bar{S}d\left(\frac{1}{E}\right) \Leftrightarrow \frac{1}{E}\bar{\sigma}\sigma_E^T\,dt\right] \\ &= \bar{\Delta}d\left(\frac{\bar{S}}{E}\right) \text{ by Itô's Lemma} \end{aligned}$$

so $\bar{\Delta}$ is self-financing with respect to $\bar{S}/E$.

Conversely, if $\bar{\Delta}$ is self-financing with respect to $\bar{S}/E$, then $1/E$ is a positive Itô process, so by what we have just proved, $\bar{\Delta}$ is self-financing with respect to $\frac{\bar{S}/E}{1/E} = \bar{S}$. ∎

Proposition 18.4 *Suppose E is a positive Itô process. Π is a state price process for $\bar{S}$ if and only if ΠE is a state price process for $\bar{S}/E$. A self-financing trading strategy $\bar{\Delta}$ is admissible for $\bar{S}$ and Π if and only if $\bar{\Delta}$ is admissible for $\bar{S}/E$ and ΠE.*

Proof:

$$\Pi E\left(\frac{\bar{S}}{E}\right) = \Pi\bar{S}$$

so there is no drift on the left side of the equation if and only if there is no drift on the right side of the equation.

$$\Pi\bar{\Delta}\bar{S} = \Pi E\bar{\Delta}\frac{\bar{S}}{E}$$

so the left side of the equation is a martingale if and only if the right side of the equation is a martingale. ∎

Remark 18.5 1. Changing the unit of account changes the state price process, hence changes the interest rate r and the vector of prices of risk λ.

2. If E has zero dispersion, λ does not change but r does. In particular, if $E = M$, the value process of a money-market account, the change from $\bar{S}$ to $\bar{S}/E$ sets the interest rate r to zero, but leaves λ unchanged.

3. Suppose $\bar{b}$ is self-financing, and $\bar{b}\bar{S} > 0$. Set $E = \bar{b}\bar{S}$. We saw that $\bar{\Delta}$ is self-financing with respect to $\bar{S}$ if and only if $\bar{\Delta}$ is self-financing with respect to $\bar{S}/E$. In particular, $\bar{b}$ is self-financing with respect to $\bar{S}/E = \bar{S}/\bar{b}\bar{S}$. The value process is

$$V(t) = \frac{\bar{b}\bar{S}}{E} = \frac{\bar{b}\bar{S}}{\bar{b}\bar{S}} = 1$$

so $\bar{S}/\bar{b}\bar{S}$ is normalized, with $\bar{b}$ as numeraire.

4. Suppose $\bar{b}$ is a money-market account with value process $M = \bar{b}\bar{S}$ and interest rate $r = \frac{\bar{b}\bar{\mu}}{\bar{b}\bar{S}}$. Take $E = M$. Notice that the coefficient of dW in $d(1/M)$ is zero, so by Itô's Lemma,

$$d\left(\frac{\bar{S}}{M}\right) = \frac{1}{M}\,d\bar{S} \Leftrightarrow \bar{S}\frac{r}{M}\,dt = \frac{1}{M}\left[(\bar{\mu} \Leftrightarrow r\bar{S})\,dt + \bar{\sigma}\,dW\right]$$

Obviously, $d(M/M) = d(1) = 0$; we can also see that from

$$\begin{aligned}
d\left(\frac{M}{M}\right) &= d\left(\frac{\bar{b}\bar{S}}{M}\right) \\
&= \bar{b}d\left(\frac{\bar{S}}{M}\right) \text{ since } \bar{b} \text{ is self-financing} \\
&= \frac{1}{M}\left[(\bar{b}\bar{\mu} \Leftrightarrow r\bar{b}\bar{S})\,dt + \bar{b}\bar{\sigma}\,dW\right] \\
&= \frac{1}{M}\left[(\bar{b}\bar{\mu} \Leftrightarrow \frac{\bar{b}\bar{\mu}}{\bar{b}\bar{S}}\bar{b}\bar{S})\,dt\right] \text{ since } \bar{b}\bar{\sigma} = 0 \\
&= 0
\end{aligned}$$

5. The largest class of stochastic processes for which stochastic integrals are defined is the class of semi-martingales. Roughly speaking, these

are processes of the form

$$S(\omega,t)=S(\omega,0)+\int_0^t a(\omega,t)\,ds+M(\omega,t)$$

where $a\in\mathcal{L}^1$ and M is "almost a martingale" with a well-behaved quadratic variation. This allows some more freedom in changing the units of account. As long as the stochastic integral is defined, the change from $\bar{S}$ to $(\bar{S}/E)$ will work fine.

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