Bus Ad 239B–Spring 2003
Solutions to Problem Set 9

There is one flaw in the argument, in the assertion that $\bar{\Delta}$ is self-financing. This appears to be true, because we tend to think of the Itô integral as a Stieltjes Integral taken pathwise; viewed that way, it looks as if $\bar{\Delta}$ should be self-financing because trade only occurs at nodes (ω, t) for which $S(\omega, t) = X$, and at those nodes the trade balances the budget.

However, recall that the Itô Integral is defined first for simple functions, then extended to $\mathcal{H}^2$ (and eventually $\mathcal{L}^2$, but that is not relevant here because $\bar{\Delta} \in \mathcal{H}^2$) by the Itô Isometry. So to analyze whether $\bar{\Delta}$ is self-financing, we need to approximate it with simple functions, compute the Stieltjes Integral of those simple functions, then take limits.

For simplicity, assume $T = 1$ and let $n \in \mathbf{N}$.

$$\bar{\Delta}(\omega, t) = \begin{cases} (-X, 1) & \text{if} \quad S(\omega, t) \geq X \\ (0, 0) & \text{if} \quad S(\omega, t) < X \end{cases}$$

Define

$$\bar{\Delta}_n(\omega, t) = \begin{cases} (-X, 1) & \text{if} \quad S\left(\omega, \frac{\lfloor nt \rfloor}{n}\right) \geq X \\ (0, 0) & \text{if} \quad S\left(\omega, \frac{\lfloor nt \rfloor}{n}\right) < X \end{cases}$$

$\bar{\Delta}_n$ is simple and adapted.

$$\begin{aligned}
&\lim_{n \to \infty} \|\bar{\Delta}_n - \bar{\Delta}\|_2^2 \\
&= \lim_{n \to \infty} \int_{\Omega \times [0,1]} (\bar{\Delta}_n(\omega, t) - \bar{\Delta}(\omega, t))^2 \, d(P \times \lambda) \\
&= \lim_{n \to \infty} \int_{\Omega} \int_{[0,1]} (\bar{\Delta}_n(\omega, t) - \bar{\Delta}(\omega, t))^2 \, d\lambda \, dP \\
&\quad \text{by Fubini's Theorem} \\
&= \int_{\Omega} \lim_{n \to \infty} \int_{[0,1]} (\bar{\Delta}_n(\omega, t) - \bar{\Delta}(\omega, t))^2 \, d\lambda \, dP \\
&\quad \text{by the Dominated Convergence Theorem} \\
&= \int_{\Omega} \int_{[0,1]} \lim_{n \to \infty} (\bar{\Delta}_n(\omega, t) - \bar{\Delta}(\omega, t))^2 \, d\lambda \, dP \\
&\quad \text{by the Dominated Convergence Theorem} \\
&= \int_{\Omega} \int_{[0,1]} \lim_{n \to \infty} 0 \, d\lambda \, dP \\
&\quad \text{since } S \text{ is continuous and } \bar{\Delta}(\omega, t) \neq X \text{ for almost all } \omega
\end{aligned}$$

Obviously, $\bar{\Delta}_n$ is not self-financing, since trades occur at fixed times k/n at which the stock price S will approximately equal X, but not exactly equal it. More to the point, $\bar{\Delta}_n$ is not approximately self-financing in the limit as $n \to \infty$. It is easier to see this if we change the problem in two ways: first, let the stock price be Brownian motion, rather than geometric Brownian motion; second, replace Brownian motion by the discrete random walk of the first lecture. In that case, the random walk $X_n\left(\omega, \frac{k}{n}\right)$ only takes on values of the form $\frac{\ell}{\sqrt{n}}$; since there are n time nodes, the stock price will equal the exercise price at roughly $n/\sqrt{n} = \sqrt{n}$ times. At each such node, with probability $1/2$, the random walk moves up at the next step, in which case no trade occurs; with probability $1/2$, the random walk moves down at the next step, in which case the stock price has moved below the exercise price, and the strategy requires selling the stock and closing out the borrowing in the money-market account. However, the value of the stock is $1/\sqrt{n}$ below the exercise price X, which equals the amount borrowed in the money-market account, so there is an expected loss of $\frac{1}{2\sqrt{n}}$ at each such node. Since there are of order $\sqrt{n}$ such nodes, the loss will be of order $\frac{\sqrt{n}}{2\sqrt{n}} = \frac{1}{2}$, which obviously does not go to zero as $n \to \infty$. Thus, $\bar{\Delta}$ is not self-financing.

You might think that the Law of Large Numbers would come to the rescue here: if the strategy loses $\frac{1}{2}$ for almost all paths ω, one simply has to invest $\frac{1}{2}$ in the money-market account at time zero and use it to cover the losses as they occur. However, the loss depends in an important way on the path ω, and specifically how much time $S(\omega, t)$ spends close to the strike price X. For some ω, $S(\omega, t) < X$ for all $t \in [0, 1]$; for other ω, $S(\omega, t)$ passes cleanly through X and stays far above X for most times; while for other ω, $S(\omega, t)$ hovers in the neighborhood of X, so the strategy $\bar{\Delta}$ piles up trading losses. Thus, the strategy $\bar{\Delta}$ cannot be made self-financing in this way.

Some people said that (8) shows that $\bar{\Delta}\bar{S}$ is not a martingale, while (7) asserts that it is a martingale, so that was a flaw. It is true that (7) and (8) contradict each other, but you need to figure out where the error that led to this contradiction occurred. The problem is the one indicated above, namely that $\bar{\Delta}$ is not self-financing. $\int \bar{\Delta} dS = \int \bar{\Delta}\bar{\sigma}\, dW$ is a martingale because $\bar{\Delta}\bar{\sigma} \in \mathcal{H}^2$. However, because $\bar{\Delta}$ is not self-financing,

$$\bar{\Delta}\bar{S} \neq \int \bar{\Delta}\, dS$$

Indeed, it turns out that $\bar{\Delta}\bar{S}$ is not an Itô process.