Bus Ad 239B–Spring 2003
Solutions to Problem Set 8

1. Let $\bar{b}$ be a trading strategy.

$$\begin{aligned}
&\bar{b}(\omega,t)\bar{S}(\omega,t) \\
&\quad = \bar{b}(\omega,0)\bar{S}(\omega,0) + \int_0^t \bar{b}(\omega,s)\bar{\mu}(\omega,s)\,ds + \int_0^t \bar{b}(\omega,s)\bar{\sigma}(\omega,s)\,dW(\omega,s)
\end{aligned}$$

Thus, if the value process $\bar{b}\bar{S}$ is instantaneously riskless, then $\bar{b}(\omega,s)\bar{\sigma}(\omega,s)$ must be zero almost everywhere. $\bar{\sigma}(\omega,s)$ is an $(N+1)\times K$-dimensional process. The rank of $(\bar{\sigma}(\omega,s))$ is N almost everywhere; if we fix (ω,s) where this holds, then

$$\{b \in \mathbf{R}^{N+1} : b\bar{\sigma}(\omega,s) = 0\} = \mathbf{R}v$$

for some $v \in \mathbf{R}^{N+1}$. Since the components of σ are measurable, adapted processes, we may choose $v(\omega,s)$ a measurable, adapted $1\times(N+1)$ process such that $|v(\omega,s)| = 1$, $v(\omega,s)\bar{S}(\omega,s) \geq 0$ and $v(\omega,s)\bar{\sigma}(\omega,s) = 0$ almost surely ($v(\omega,s)$ is a unit vector in the space perpendicular to the span of the columns of $\bar{\sigma}(\omega,s)$). Thus, we must have $\bar{b}(\omega,s) = \alpha(\omega,s)v(\omega,s)$ for some scalar process α.

$\bar{b} = \alpha v$ will be a money-market account provided that $\bar{b} \in \mathcal{L}(\bar{S})$ (i.e. $\bar{b}\bar{\mu} = \alpha v\bar{\mu} \in \mathcal{L}^1$), $\bar{b}$ is self-financing, and almost surely in ω, $M(\omega,t) = \bar{b}(\omega,t)\bar{S}(\omega,t) > 0$ for all t. Since $M = \bar{b}\bar{S} = \alpha v\bar{S}$, we will be in serious trouble if $v\bar{S} = 0$. Hence, we propose as our first condition that

$$P\left(\left\{\omega : v(\omega,t)\bar{S}(\omega,t) = 0 \text{ for some } t\right\}\right) = 0 \ (*)$$

or equivalently,

$$P\left(\left\{\omega : \bar{S}(\omega,t) \in \text{span}\{\bar{\sigma}_1,\ldots\bar{\sigma}_K\} \text{ for some } t\right\}\right) = 0$$

where $\bar{\sigma}_k$ denotes the k^{th} column of $\bar{\sigma}$. The interest rate on the money-market acount must be

$$\frac{\bar{b}(\omega,s)\bar{\mu}(\omega,s)}{\bar{b}(\omega,s)\bar{S}(\omega,s)} = \frac{\alpha(\omega,s)v(\omega,s)\bar{\mu}(\omega,s)}{\alpha(\omega,s)v(\omega,s)\bar{S}(\omega,s)} = \frac{v(\omega,s)\bar{\mu}(\omega,s)}{v(\omega,s)\bar{S}(\omega,s)}$$

Thus, the value process of the money-market account must (up to a scalar multiple) have the form

$$M(\omega, t) = e^{\int_0^t \frac{v(\omega,s)\bar{\mu}(\omega,s)}{v(\omega,s)\bar{S}(\omega,s)}\, ds}$$

For this definition to make sense, we will need to know that

$$\frac{v(\omega, s)\bar{\mu}(\omega, s)}{v(\omega, s)\bar{S}(\omega, s)} \in \mathcal{L}^1$$

Since we have already assumed in (*) that almost surely in ω, $v(\omega, s)\bar{S}(\omega, s) > 0$ for all s, let us also assume that

$$P\left(\{\omega : \bar{\sigma}(\omega, t) \text{ is continuous in } t \text{ and } \operatorname{rank}(\bar{\sigma}(\omega, t)) = N \text{ for all } t\}\right) = 1 \quad (**)$$

since this implies that $v(\omega, s)$ is continuous.

We now show that $(*)$ and $(**)$ imply that there is a money-market account. Notice that $\bar{\mu} \in \mathcal{L}^1$ by the definition of $\bar{S}$, and for almost all ω, $\frac{|\bar{v}(\omega,s)|}{v(\omega,s)\bar{S}(\omega,s)}$ is bounded over bounded time intervals $[0, T]$, so

$$\frac{v(\omega, s)\bar{\mu}(\omega, s)}{v(\omega, s)\bar{S}(\omega, s)} \in \mathcal{L}^1$$

as desired. Let

$$\alpha(\omega, s) = \frac{M(\omega, s)}{v(\omega, s)\bar{S}(\omega, s)}$$

By Itô's Lemma, $M \in \mathcal{L}^1$ and by (*) and (**), $v(\omega, s)\bar{S}(\omega, s)$ is almost surely bounded away from zero over finite time intervals $[0, T]$, so $b(\omega, s) = \alpha(\omega, s)v(\omega, s) \in \mathcal{L}^1$. Then

$$\begin{aligned}
&\bar{b}(\omega, s)\bar{S}(\omega, 0) \\
&\quad = \alpha(\omega, s)v(\omega, s)\bar{S}(\omega, s) \\
&\quad = \frac{M(\omega, s)v(\omega, s)\bar{S}(\omega, s)}{v(\omega, s)\bar{S}(\omega, s)} \\
&\quad = M(\omega, s)
\end{aligned}$$

and

$$\begin{aligned}\frac{dM}{M} &= \frac{v(\omega,s)\bar{\mu}(\omega,s)}{v(\omega,s)\bar{S}(\omega,s)}\,dt \\ &= \frac{\alpha(\omega,s)v(\omega,s)\bar{\mu}(\omega,s)}{\alpha v(\omega,s)\bar{S}(\omega,s)}\,dt \\ &= \frac{\bar{b}\,d\bar{S}}{M}\end{aligned}$$

so

$$d(\bar{b}\bar{S}) = \bar{b}\,d\bar{S}$$

which says that $\bar{b}$ is self-financing.

2. For now, fix (ω, t). We form an orthonormal basis V of $\mathbf{R}^{N+1}$. The zeroth element is $v_0 = \bar{b}(\omega,t)/|\bar{b}(\omega,t)|$. Use the Gram-Schmidt process to fill in the remaining basis elements $v_2, \ldots, v_N$. Let

$$B = \begin{pmatrix} v_0 & v_1 & \cdots & v_N \\ | & | & & | \end{pmatrix}$$

i.e. the n^{th} column of B is v_n. B is a unitary matrix, so $B^{-1} = B^T$. Let

$$\hat{S} = B^{-1}\bar{S},\ \hat{\mu} = B^{-1}\bar{\mu},\ \hat{\sigma} = B^{-1}\bar{\sigma}$$

$\lambda \in \mathcal{L}^2$ is a vector of prices of risk for $\bar{S}$ and r if and only if

$$\bar{\mu} - r\bar{S} = \bar{\sigma}\,(\lambda)^T$$

if and only if

$$B\hat{\mu} - rB\hat{S} = B\hat{\sigma}\,(\lambda)^T$$

if and only if λ is a vector of prices of risk for $\hat{S}$ and r.

The stochastic integral equation for $\hat{S}$ is

$$\hat{S}(t) = \hat{S}(0) + \int_0^t \hat{\mu}\,ds + \int_0^t \hat{\sigma}\,dW$$

The zeroth row of $\hat{\sigma}$ is identically zero, the zeroth security is a money market fund with interest rate r, and $\hat{\sigma}$ has rank N. Define S, $\tilde{\mu}$, and

$\tilde{\sigma}$ from $\hat{S}$, $\hat{\mu}$, and $\hat{\sigma}$ in the same way as Nielsen derives them from $\bar{S}$, $\bar{\mu}$, and $\bar{\sigma}$ on page 135. Let

$$\lambda^* = (\tilde{\mu} - rS)^T \left(\tilde{\sigma}\tilde{\sigma}^T\right)^{-1} \tilde{\sigma}$$

By Proposition 4.5 of Nielsen, there exists a vector of prices of risk for $\hat{S}$ and r if and only if $\lambda^* \in \mathcal{L}^2$, in which case λ^* is the minimal vector of prices of risk for S and r. Thus, there exists a vector of prices of risk for $\bar{S}$ and r if and only if $\lambda^* \in \mathcal{L}^2$, in which case λ^* is the minimal vector of prices of risk for $\bar{S}$ and r.