Bus Ad 238B–Spring 2002
Solutions to Problem Set 7

1. Let $b \in \mathcal{L}(\bar{S})$. By Itô's Lemma,

$$d\bar{S} = \bar{S}\bar{\mu}\,dt + \bar{S}\bar{\sigma}\,dW$$

so

$$\int b d\bar{S} = \int b(\bar{S}\bar{\mu})\,dt + \int b(\bar{S}\bar{\sigma})\,dW$$

Note that when we write the product of two column vectors such as $\bar{S}$ and $\bar{\mu}$, the product is taken componentwise, i.e.

$$\bar{S}\bar{\mu} = \begin{pmatrix} \bar{S}_0 \\ \bar{S}_1 \\ \bar{S}_2 \end{pmatrix} \begin{pmatrix} \bar{\mu}_0 \\ \bar{\mu}_1 \\ \bar{\mu}_2 \end{pmatrix} = \begin{pmatrix} \bar{S}_0\bar{\mu}_0 \\ \bar{S}_1\bar{\mu}_1 \\ \bar{S}_2\bar{\mu}_2 \end{pmatrix}$$

similarly

$$\bar{S}\bar{\sigma} = \begin{pmatrix} \bar{S}_0 \\ \bar{S}_1 \\ \bar{S}_2 \end{pmatrix} \begin{pmatrix} \bar{\sigma}_{01} & \bar{\sigma}_{02} \\ \bar{\sigma}_{11} & \bar{\sigma}_{12} \\ \bar{\sigma}_{21} & \bar{\sigma}_{22} \end{pmatrix} = \begin{pmatrix} \bar{S}_0\bar{\sigma}_{01} & \bar{S}_0\bar{\sigma}_{02} \\ \bar{S}_1\bar{\sigma}_{11} & \bar{S}_1\bar{\sigma}_{12} \\ \bar{S}_2\bar{\sigma}_{21} & \bar{S}_2\bar{\sigma}_{22} \end{pmatrix}$$

$\int_0^t b\,d\bar{S}$ is deterministic if and only if

$$b(\omega,t)(\bar{S}(\omega,t)\bar{\sigma}) = 0$$

almost everywhere. But

$$\begin{aligned}
b(\bar{S}\bar{\sigma}) &= \begin{pmatrix} b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} \bar{S}_0\bar{\sigma}_{01} & \bar{S}_0\bar{\sigma}_{02} \\ \bar{S}_1\bar{\sigma}_{11} & \bar{S}_1\bar{\sigma}_{12} \\ \bar{S}_2\bar{\sigma}_{21} & \bar{S}_2\bar{\sigma}_{22} \end{pmatrix} \\
&= \begin{pmatrix} b_0\bar{S}_0\bar{\sigma}_{01} & b_0\bar{S}_0\bar{\sigma}_{02} \\ b_1\bar{S}_1\bar{\sigma}_{11} & b_1\bar{S}_1\bar{\sigma}_{12} \\ b_2\bar{S}_2\bar{\sigma}_{21} & b_2\bar{S}_2\bar{\sigma}_{22} \end{pmatrix} \\
&= \begin{pmatrix} b_0\bar{S}_0 & b_1\bar{S}_1 & b_2\bar{S}_2 \end{pmatrix} \begin{pmatrix} \bar{\sigma}_{01} & \bar{\sigma}_{02} \\ \bar{\sigma}_{11} & \bar{\sigma}_{12} \\ \bar{\sigma}_{21} & \bar{\sigma}_{22} \end{pmatrix} \\
&= (b(\bar{S})^T)\bar{\sigma}
\end{aligned}$$

Since $\bar{\sigma}$ has rank 2, $\ker(\bar{\sigma})$, the kernel of $\bar{\sigma}$ (i.e. the set of $v \in \mathbf{R}^3$ such that $v(\bar{\sigma}) = 0$) is 1-dimensional; thus, there is $v \in \mathbf{R}^3$ such that $\ker(\bar{\sigma}) = \mathbf{R}v$. Without loss of generality, we may assume that $|v| = 1$.

Thus, $\int_0^T b\, d\bar{S}$ is deterministic for all t if and only if there exists a scalar process α such that

$$b(\omega,t)(\bar{S}(\omega,t))^T = \alpha(\omega,t)v$$

if and only if

$$b(\omega,t) = \alpha(\omega,t)\frac{v}{(\bar{S}(\omega,t))^T}$$

almost everywhere, i.e.

$$b_n(\omega,t) = \frac{\alpha(\omega,t)v_n}{\bar{S}_n(\omega,t)}$$

almost everywhere. Notice that the last formula makes sense because $\bar{S}_n(\omega,t) > 0$ for all (ω,t).

If $b(\omega,t)\bar{S}(\omega,t) = 1$, then

$$\alpha(\omega,t)\left[\frac{v_0\bar{S}_0(\omega,t)}{\bar{S}_0(\omega,t)} + \frac{v_1\bar{S}_1(\omega,t)}{\bar{S}_1(\omega,t)} + \frac{v_2\bar{S}_2(\omega,t)}{\bar{S}_2(\omega,t)}\right] = 1$$

if and only if

$$\alpha(\omega,t) = \frac{1}{v_0+v_1+v_2} \tag{1}$$

Note the last formula makes sense, as long as the denominator is nonzero. This proves uniqueness. Assuming that $v_0+v_1+v_2 \neq 0$, we let

$$\bar{b}_0(\omega,t) = \frac{1}{v_0+v_1+v_2}\frac{v}{(\bar{S}(\omega,t))^T}$$

Since $\bar{S}_n(\omega,t) > 0$ for all (ω,t), for all ω we have $S_n(\omega,t)$ is uniformly bounded away from zero on each finite interval $[0,T]$, which shows $\bar{b}_0 \in \mathcal{L}(\bar{S})$ and establishes existence as long as $v_0+v_1+v_2 \neq 0$. Notice that the condition $v_0+v_1+v_2 \neq 0$ is a condition on $\bar{\sigma}$; it is generically satisfied, i.e. if we put 6-dimensional Lebesgue measure on the space of 3×2 matrices $\bar{\sigma}$, then except for a set of Lebesgue measure zero, $\bar{\sigma}$ has rank 2 and $v \in \ker(\bar{\sigma})$, $v \neq 0 \Rightarrow v_0+v_1+v_2 \neq 0$. For the rest of the problem, we assume that $v_0+v_1+v_2 \neq 0$.

2. $\bar{b}_0$ is self-financing if and only if

$$b_0 \, d\bar{S} = d(b_0 \bar{S}) = d(1) = 0$$

if and only if

$$\frac{v}{(\bar{S})^T}(\bar{S}\bar{\mu}) = 0$$

if and only if

$$\left(\frac{v_0}{\bar{S}_0(\omega,t)}, \frac{v_1}{\bar{S}_1(\omega,t)}, \frac{v_2}{\bar{S}_2(\omega,t)}\right) \begin{pmatrix} \bar{S}_0(\omega,t)\bar{\mu}_0 \\ \bar{S}_1(\omega,t)\bar{\mu}_1 \\ \bar{S}_2(\omega,t)\bar{\mu}_2 \end{pmatrix} = 0$$

if and only if

$$v\bar{\mu} = 0$$

Since $v \in \ker \bar{\sigma}$, $v\bar{\mu} = 0$ if and only if

$$\bar{\mu} \in \operatorname{span} \left\{ \begin{pmatrix} \bar{\sigma}_{01} \\ \bar{\sigma}_{11} \\ \bar{\sigma}_{21} \end{pmatrix}, \begin{pmatrix} \bar{\sigma}_{02} \\ \bar{\sigma}_{12} \\ \bar{\sigma}_{22} \end{pmatrix} \right\} \tag{2}$$

i.e. $\bar{\mu}$ is in the span of the two columns of $\bar{\sigma}$.

3. We saw in part (1) that $\int b \, d\bar{S}$ is deterministic if and only if

$$b(\omega,t) = \alpha(\omega,t)\frac{v}{(\bar{S}(\omega,t))^T}$$

for some scalar process α. b is self-financing if and only if

$$b\bar{S}(t) = b(0)\bar{S}(0) + \int_0^t b \, d\bar{S} \text{ for all } t$$

4. Solving the stochastic differential equation,

$$\frac{\alpha(\omega,t)v}{(\bar{S}(\omega,t))^T}\bar{S}(\omega,t) = \frac{\alpha(\omega,0)v}{(\bar{S}(\omega,0))^T}\bar{S}(\omega,0) + \int_0^t \frac{\alpha(\omega,t)v}{(\bar{S}(\omega,t))^T} \, d\bar{S}$$

$$\Leftrightarrow \quad \alpha(\omega,t)(v_0+v_1+v_2) = \alpha(\omega,0)(v_0+v_1+v_2) + \int_0^t \frac{\alpha(\omega,t)v}{(\bar{S}(\omega,t))^T}(\bar{S}\bar{\mu}) \, dt$$

$$\Leftrightarrow \quad \alpha(\omega,t) = \alpha(\omega,0) + \frac{1}{v_0+v_1+v_2}\int_0^t \alpha(\omega,t)(v\bar{\mu}) \, dt$$

$$\Leftrightarrow \quad \alpha(\omega,t) = \alpha(\omega,0)e^{(v\bar{\mu}t)/(v_0+v_1+v_2)}$$

$$\Leftrightarrow \quad b(\omega,t) = \frac{\alpha(\omega,0)e^{(v\bar{\mu}t)/(v_0+v_1+v_2)}v}{(\bar{S}(\omega,t))^T}$$

The money market fund is

$$\begin{aligned} M(\omega,t) &= b(\omega,t)\bar{S}(\omega,t) \\ &= \alpha(\omega,0)\frac{e^{(v\bar{\mu}t)/(v_0+v_1+v_2)}v}{(\bar{S}(\omega,t))^T}\bar{S}(\omega,t) \\ &= \alpha(\omega,0)(v_0+v_1+v_2)e^{(v\bar{\mu}t)/(v_0+v_1+v_2)} \end{aligned}$$

The interest rate process $r(\omega,t)$ satisfies

$$r(\omega,t)\,dt = \frac{dM}{M} = \frac{v\bar{\mu}}{v_0+v_1+v_2}\,dt$$

so

$$r(\omega,t) = \frac{v\bar{\mu}}{v_0+v_1+v_2} = \frac{v_0\bar{\mu}_0+v_1\bar{\mu}_1+v_2\bar{\mu}_2}{v_0+v_1+v_2}$$

5. Let

$$\bar{r} = \begin{pmatrix} r \\ \vdots \\ r \end{pmatrix}$$

The stochastic integral equation is

$$\bar{S}(t) = \bar{S}(0) + \int_0^t \bar{S}\bar{\mu}\,ds + \int_0^T \bar{S}\bar{\sigma}\,dW$$

Putting this into the form of Chapter 4 of Nielsen, the vector λ of prices of risk must satisfy

$$(\bar{S}\bar{\mu}) \Leftrightarrow (\bar{S}\bar{r}) = (\bar{S}\bar{\sigma})\lambda^T$$

We can divide through the equation by $\bar{S}$ (as noted in part (1), it just multiplies elements componentwise) to obtain

$$\bar{\mu} \Leftrightarrow \bar{r} = \bar{\sigma}\lambda^T$$

This equation has a solution if and only if $\bar{\mu} \Leftrightarrow \bar{r}$ lies in the span of the columns of $\bar{\sigma}$. Since rank $\bar{\sigma} = 2$, and v is perpendicular to the columns of $\bar{\sigma}$, the span of the columns of $\bar{\sigma}$ consists exactly of those vectors perpendicular to v. Notice that

$$\begin{aligned} v(\bar{\mu} \Leftrightarrow \bar{r}) &= v_0\bar{\mu}_0 + v_1\bar{\mu}_1 + v_2\bar{\mu}_2 \Leftrightarrow [v_0 r + v_1 r + v_2 r] \\ &= v_0\bar{\mu}_0 + v_1\bar{\mu}_1 + v_2\bar{\mu}_2 \Leftrightarrow [v_0 + v_1 + v_2]\, r \\ &= v_0\bar{\mu}_0 + v_1\bar{\mu}_1 + v_2\bar{\mu}_2 \Leftrightarrow [v_0 + v_1 + v_2] \left(\frac{v_0\bar{\mu}_0 + v_1\bar{\mu}_1 + v_2\bar{\mu}_2}{v_0 + v_1 + v_2} \right) \\ &= v_0\bar{\mu}_0 + v_1\bar{\mu}_1 + v_2\bar{\mu}_2 \Leftrightarrow (v_0\bar{\mu}_0 + v_1\bar{\mu}_1 + v_2\bar{\mu}_2) \\ &= 0 \end{aligned}$$

so the equation $\bar{\mu} \Leftrightarrow \bar{r} = \bar{\sigma}\lambda^T$ has a solution. Indeed, if we let

$$\lambda = (\bar{\mu} \Leftrightarrow \bar{r})^T (\bar{\sigma}\bar{\sigma}^T)^{-1} \bar{\sigma}$$

then

$$\begin{aligned} \bar{\sigma}\lambda^T &= \bar{\sigma}\bar{\sigma}^T \left((\bar{\sigma}\bar{\sigma}^T)^{-1} \right)^T (\bar{\mu} \Leftrightarrow \bar{r}) \\ &= (\bar{\sigma}\bar{\sigma}^T)^{-1} (\bar{\sigma}\bar{\sigma}^T) (\bar{\mu} \Leftrightarrow \bar{r}) \\ &= \bar{\mu} \Leftrightarrow \bar{r} \end{aligned}$$

The state price proces is

$$\Pi(t) = \Pi(0)\eta[\Leftrightarrow r, \Leftrightarrow \lambda]$$