Bus Ad 239B–Spring 2003
Solutions to Problem Set 6

1. We pick up from the calculation in the Lecture Notes. There, we found that if the value of the discrete Brownian bridge at time k/n is

$$B\left(\omega, \frac{k}{n}\right) = \frac{\ell}{\sqrt{n}}$$

$$P(\omega_{k+1} = +1) = \tfrac{nT-k-\ell}{2(nT-k)} \quad P(\omega_{k+1} = -1) = \tfrac{nT-k+\ell}{2(nT-k)}$$

and

$$E\left(\Delta B\left(\omega, \frac{k}{n}\right)\right) = E\left(B\left(\omega, \frac{k+1}{n}\right) - B\left(\omega, \frac{k}{n}\right)\right) = \frac{-\ell}{\sqrt{n}(nT-k)}$$

Therefore,

$$\begin{aligned}
\sigma^2(\omega,k,n) &= \mathrm{Var}\left(\Delta B\left(\omega, \frac{k}{n}\right)\right) \\
&= \frac{nT-k-\ell}{2(nT-k)}\left[\frac{1}{\sqrt{n}} - \frac{-\ell}{\sqrt{n}(nT-k)}\right]^2 + \frac{nT-k+\ell}{2(nT-k)}\left[-\frac{1}{\sqrt{n}} - \frac{-\ell}{\sqrt{n}(nT-k)}\right]^2 \\
&= \frac{nT-k-\ell}{2(nT-k)}\left[\frac{1}{n} + \frac{2\ell}{n(nT-k)} + \frac{\ell^2}{n(nT-k))^2}\right] \\
&\quad + \frac{nT-k+\ell}{2(nT-k)}\left[\frac{1}{n} - \frac{2\ell}{n(nT-k)} + \frac{\ell^2}{n(nT-k))^2}\right] \\
&= \frac{(nT-k-\ell)+(nT-k+\ell)}{2(nT-k)}\left[\frac{1}{n} + \frac{\ell^2}{n(nT-k))^2}\right] \\
&\quad + \frac{nT-k}{2(nT-k)}\left[\frac{2\ell-2\ell}{n(nT-k)}\right] - \frac{4\ell^2}{2n(nT-k)^2} \\
&= \left[\frac{1}{n} + \frac{\ell^2}{n(nT-k))^2}\right] - \frac{2\ell^2}{n(nT-k)^2} \\
&= \frac{1}{n} - \frac{\ell^2}{n(nT-k)^2}
\end{aligned}$$

As $n \to \infty$ and $k_n/n \to t \in [0, T)$,

$$\begin{aligned}
\lim_{n\to\infty} n\sigma^2(\omega, k_n, n) &= \lim_{n\to\infty} \left(1 - \frac{\ell^2}{(nT - k_n)^2}\right) \\
&= \lim_{n\to\infty} \left(1 - \left(\frac{\ell}{\sqrt{n}}\right)^2 \frac{1}{(T-t)^2}\frac{1}{n}\right) \\
&= 1
\end{aligned}$$

in distribution, since $\frac{\ell}{\sqrt{n}(T-t)} = \frac{B(\omega, k_n/n)}{T-t}$ converges in distribution. If $k_n/n \to T$ sufficiently slowly, since $B(\omega, k_n/n)$ has the same distribution as $B(\omega, T - k_n/n)$, ℓ will be of order $\sqrt{nT - k_n}$ so

$$\frac{\ell^2}{(nT - k_n)^2} \to 0$$

in distribution and $n\sigma^2(\omega, k_n, n) \to 1$ in distribution. On the other hand, if $k_n/n \to T$ rapidly, $n\sigma^2(\omega, k_n, n)$ will be somewhere between 0 and 1. For example, if $k_n = nT - 1$, then $\ell = \pm 1$ for sure and $n\sigma^2(\omega, k_n, n) = 0$; this makes sense because, the last time you draw from the urn, there is only one ball left, so there is no uncertainty.

These facts give strong evidence that the stochastic differential equation defining the continuous Brownian bridge must be

$$dB(\omega, t) = -\frac{B(\omega, t)}{T - t} + dW(\omega, t)$$

It is possible to turn this into a rigorous proof.