

Bus Ad 239B–Spring 2003 Solutions to Problem Set 5

1. (a) i. Let

$$\begin{aligned} Z(\omega, t) &= W(\omega, t)^2 \\ &= 2 \int_0^t W dW + t \end{aligned}$$

$$\begin{aligned} &\int_{\Omega \times [0, t]} W(\omega, s)^2 d(P \otimes \lambda) \\ &= \int_0^t \left(\int_{\Omega} W(\omega, s)^2 dP \right) d\lambda \\ &= \int_0^t s ds \\ &= \frac{t^2}{2} < \infty \end{aligned}$$

so $W \in \mathcal{H}^2$, so $\int_0^t W dW$ is a martingale. Thus, if we let $M(\omega, t) = 2 \int_0^t W dW$ and $A(\omega, t) = t$, then $Z = M + A$, M is a martingale and A is a process of bounded variation over bounded time intervals.

ii. Let

$$\begin{aligned} Z(\omega, t) &= e^{(\mu-2)t+2W(\omega, t)} \\ &= 1 + (\mu - 2 + 2^2/2) \int_0^t Z(\omega, s) ds + 2 \int_0^t Z(\omega, s) dW(\omega, s) \\ &= 1 + \mu \int_0^t Z(\omega, s) ds + 2 \int_0^t Z(\omega, s) dW(\omega, s) \end{aligned}$$

Let $A(\omega, t) = 1 + \mu \int_0^t Z(\omega, s) ds$ and $M(\omega, t) = 2 \int_0^t Z(\omega, s) dW(\omega, s)$. $|Z(\omega, \cdot)|$ is continuous almost surely, so assumes its maximum over $[0, t]$; call that maximum $\alpha(\omega, t)$. Then we see that $A(\omega, \cdot)$ is Lipschitz on $[0, t]$ with Lipschitz constant $\mu\alpha(\omega, t)$, hence $A(\omega, \cdot)$ is of bounded variation on $[0, t]$ almost surely.

$$\int_{\Omega \times [0, t]} Z(\omega, s)^2 d(P \otimes \lambda)$$

$$\begin{aligned}
&= \int_0^t e^{2(\mu-2)s} \left(\int_{\Omega} e^{4W(\omega,s)} dP \right) ds \\
&= \int_0^t e^{2(\mu-2)s} e^{8s} ds \\
&\quad (\text{expectation of a log normal}) \\
&= \int_0^t e^{(2\mu+4)s} ds \\
&= \frac{e^{(2\mu+4)t} - 1}{2\mu + 4} < \infty
\end{aligned}$$

Note that if $2\mu + 4 = 0$, the integral is just t . Thus, $Z \in \mathcal{H}^2$, so M is a martingale.

- (b) i. If $M(\omega, t) = 2 \int_0^t W dW = \int_0^t 2W dW$, let

$$\tau(\omega, t) = \int_0^t (2W(\omega, s))^2 ds = 4 \int_0^t W(\omega, s)^2 ds$$

For almost all ω , $W(\omega, s) \neq 0$ for almost all s , so $\tau(\omega, \cdot)$ is almost surely strictly increasing at every time t , so $\tau^{-1}(\omega, t)$ is defined and $\tau(\omega, \tau^{-1}(\omega, t)) = t$. Let

$$\hat{M}(\omega, t) = M(\omega, \tau^{-1}(\omega, t))$$

By Lévy's Theorem, \hat{M} is a standard Brownian motion. $M(\omega, t) = \hat{M}(\omega, \tau(\omega, t))$ so M is a standard Brownian motion with a time change.

- ii. If

$$M(\omega, t) = 2 \int_0^t Z(\omega, s) dW(\omega, s) = \int_0^t 2Z(\omega, s) dW(\omega, s)$$

let

$$\tau(\omega, t) = \int_0^t (2Z(\omega, s))^2 ds = 4 \int_0^t Z(\omega, s)^2 ds$$

$Z(\omega, s)^2 > 0$ for all (ω, s) , so $\tau(\omega, \cdot)$ is strictly increasing for all ω , so $\tau^{-1}(\omega, t)$ is defined and $\tau(\omega, \tau^{-1}(\omega, t)) = t$. Let

$$\hat{M}(\omega, t) = M(\omega, \tau^{-1}(\omega, t)) - 1$$

By Lévy's Theorem, \hat{M} is a standard Brownian motion. $M(\omega, t) = \hat{M}(\omega, \tau(\omega, t))$, so M is a standard Brownian motion with a time change.

- (c) i. In the first case, we saw in part (a) that

$$Z(\omega, t) = \int_0^t 1 ds + \int_0^t 2W(\omega, s) dW(\omega, s)$$

In the version of Girsanov's Theorem in the Lecture Notes, we are told to take

$$\lambda(\omega, s) = \frac{1}{2W(\omega, s)}$$

and let Q be the measure whose Radon-Nikodym derivative with respect to P is

$$\eta[0, -\lambda](T)$$

This works fine as long as $W(\omega, s) \neq 0$. In particular, at any (ω, s) such that $W(\omega, s) \neq 0$, we can find a probability measure Q such that Z is a martingale locally around (ω, s) . You might hope that this would work globally, since the set of (ω, s) such that $W(\omega, s) = 0$ is a set of measure zero, and so it might still be possible to integrate everything out. However, notice that

$$\eta[0, -\lambda](T) = e^{-\frac{1}{8} \int_0^T \frac{1}{W^2(\omega, s)} ds + \int_0^T \frac{1}{2W(\omega, s)} dW(\omega, s)}$$

$W(\omega, s)$ is of order \sqrt{s} , $1/W^2(\omega, s)$ is of order $1/s$, so $\int_0^T (1/W^2(\omega, s)) ds$ will look something like $\int_0^T 1/s ds$ which diverges; for similar reasons, $\int_0^T (1/2W(\omega, s)) dW(\omega, s)$ diverges.

To get a sense of what is going on here, look again at the equation

$$Z(\omega, t) = \int_0^t 1 ds + \int_0^t 2W(\omega, s) dW(\omega, s)$$

At nodes (ω, s) such that $W(\omega, s) = 0$, the instantaneous covariance of Z is zero. The discrete-case proof of Girsanov's Theorem depended on being able to make small changes in the probabilities in order to cancel out the drift term. In the random walk, the steps are of order $1/\sqrt{n}$, which are much bigger than $dt = 1/n$; it is that fact that makes the changes in

probability small. From the Law of the Iterated Logarithm, it follows that

$$\limsup_{t \searrow 0} \frac{W^2(\omega, t)}{2t \log |\log t|} = 1 \text{ and } \liminf_{t \searrow 0} \frac{W^2(\omega, t)}{2t \log |\log t|} = 0$$

In other words, for t small, Z is between 0 and something not a whole lot bigger than $2t$, while $E(Z(t)) = t$. There simply isn't enough volatility in Z to let us absorb the drift into the probabilities. Fortunately for Finance, stocks are in practice quite volatile, so it is reasonable to model the stock price process as the exponential of a stochastic integral, where the instantaneous covariance of the stochastic integral is not zero.

ii. We found in part (a) that

$$Z(\omega, t) = 1 + \int_0^t \mu Z(\omega, s) ds + \int_0^t 2Z(\omega, s) dW(\omega, s)$$

We take

$$\begin{aligned} \lambda(\omega, s) &= \frac{\mu Z(\omega, s)}{2Z(\omega, s)} = \frac{\mu}{2} \\ \eta[0, -\lambda](T) &= e^{\int_0^T \left(-\frac{1}{2} \times \frac{\mu^2}{4}\right) ds - \int_0^T \frac{\mu}{2} dW} \\ &= e^{-\mu W(\omega, T)/2 - \mu^2 T/8} \end{aligned}$$

The Novikov Condition is obviously satisfied, so if we let Q be the probability measure with Radon-Nikodym derivative

$$e^{-\mu W(\omega, T)/2 - \mu^2 T/8}$$

then Z is a martingale with respect to Q .

- (d) You might have guessed the answer was yes. Consider an n -step random walk, where each step is normally distributed with given mean and variance, instead of just taking on two values. You can adjust the probabilities at each step to set the instantaneous mean and variance to whatever values you want, in particular setting the instantaneous mean to zero and the instantaneous variance to 1.

For each n , the probability Q_n that you obtain is equivalent to the original measure P_n . Then take limits as $n \rightarrow \infty$; if Q is the limit of the measures Q_n , Q is the measure of a standard Wiener process. The problem with this approach is that P_n and Q_n need to concentrate more and more of their weight on disjoint sets, so the limit measures P and Q are mutually singular.

Thus, the answer is negative: probability adjustments cannot be used to change variances. Here is an easy way to see that. As we saw in part (b), by Levy's Theorem,

$$\hat{M}(\omega, t) = M(\omega, \tau^{-1}(\omega, t))$$

is a standard Wiener process. By the Law of the Iterated Logarithm,

$$\limsup_{\varepsilon \searrow 0} \frac{\hat{M}(\omega, t + \varepsilon) - \hat{M}(\omega, t)}{\sqrt{2\varepsilon \log \log \varepsilon}} = 1$$

so

$$\limsup_{\varepsilon \searrow 0} \frac{M(\omega, \tau^{-1}(t + \varepsilon)) - M(\omega, \tau^{-1}(t))}{\sqrt{2\varepsilon \log \log \varepsilon}} = 1$$

Unless $\tau^{-1}(t + \varepsilon) \simeq \tau^{-1}(t) + \varepsilon$, the paths of M will almost surely obey an Iterated Logarithm Law different from the Iterated Logarithm Law obeyed by a standard Wiener process, hence if Q is a probability measure that makes M into a standard Wiener process, P and Q must be mutually singular. To take a simple example, if $M = 2W$,

$$\tau(\omega, t) = 4t \text{ and } \tau^{-1}(\omega, t) = \frac{t}{4}$$

so

$$\tau^{-1}(t + \varepsilon) - \tau^{-1}(t) = \frac{\varepsilon}{4}$$

so

$$\limsup_{\varepsilon \searrow 0} \frac{M\left(\omega, \tau^{-1}(t) + \frac{\varepsilon}{4}\right) - M(\omega, \tau^{-1}(t))}{\sqrt{2\varepsilon \log \log \varepsilon}} = 1$$

Letting

$$\delta = \frac{\varepsilon}{4}$$

$$\begin{aligned}
& \limsup_{\delta \searrow 0} \frac{M(\omega, \tau^{-1}(t) + \delta) - M(\omega, \tau^{-1}(t))}{\sqrt{2\delta \log \log \delta}} \\
&= \lim_{\delta \searrow 0} \frac{\sqrt{8\delta \log \log 4\delta}}{\sqrt{2\delta \log \log \delta}} \\
&= 2
\end{aligned}$$

P -almost surely; but for any Q that makes M a standard Wiener process, the lim sup is 1 Q -almost surely. If we let B be the set where the lim sup is 2 and C be the set where the lim sup is 1, then $B \cap C = \emptyset$ and $P(B) = 1$, $Q(C) = 1$, so P and Q are mutually singular.