Bus Ad 239B–Spring 2003
Solutions to Problem Set 4

1.

$$\begin{aligned}(W(t))^{n+1} &= \left(\int_0^t dW\right)^{n+1} \\ &= \int_0^t \left((n+1)W^n \times 0 + \frac{n(n+1)}{2}W^{n-1}\right) ds \\ &\quad + \int_0^t (n+1)W^n\, dW \\ &= \frac{n(n+1)}{2}\int_0^t W^{n-1}\, ds + (n+1)\int_0^t W^n dW \qquad (1)\end{aligned}$$

so

$$\begin{aligned}&\int_0^t (W(\omega,s))^n\, dW(\omega,s) \\ &= \frac{W(\omega,t)^{n+1}}{n+1} - \frac{n}{2}\int_0^t (W(\omega,s))^{n-1}\, ds\end{aligned}$$

Taking $n=2$ in Equation (1), we have

$$\int_0^t W(\omega,s)\, ds = \frac{(W(\omega,t))^3}{3} - \int_0^t (W(\omega,s))^2\, dW$$

Alternatively, use Integration by parts (Proposition 2.15 in Nielsen) to obtain

$$\begin{aligned}\int_0^t W(\omega,s)\, ds &= W(\omega,t)t - W(\omega,0)0 - \int_0^t s\, dW(\omega,s) \\ &= tW(\omega,t) - \int_0^t s\, dW(\omega,s) \\ &= \int_0^t t\, dW(\omega,s) - \int_0^t s dW(\omega,s) \\ &= \int_0^t (t-s) dW(\omega,s)\end{aligned}$$

2. I've written these out in gory detail, to help in case you have problems with the calculations. Shorter solutions are perfectly acceptable; indeed, they are preferable.

(a) $Z(\omega,t) = e^{(W_1(\omega,t))^2} = e^{(\int_0^t 0ds + \int_0^t 1dW_1)^2}$. We can treat W_1 as a 1-dimensional Wiener process. Let $g(w) = e^{w^2}$, then $g'(w) = 2we^{w^2}$ and $g''(w) = 2e^{w^2} + 2we^{w^2}(2w) = (4w^2+2)e^{w^2}$. Thus,

$$\begin{aligned}
Z(\omega,t) &= Z(\omega,0) + \int_0^t \left(g'(W_1(\omega,s)) \times 0 + \frac{1}{2}g''(W_1(\omega,s)) \times 1\right) ds \\
&\quad + \int_0^t (g'(W_1(\omega,s)) \times 1)\, dW_1(\omega,s) \\
&= 1 + \frac{1}{2}\int_0^t (4(W_1(\omega,s))^2 + 2)e^{(W_1(\omega,s))^2}\, ds \\
&\quad + \int_0^t 2W_1(\omega,s)e^{(W_1(\omega,s))^2}\, dW_1 \\
&= 1 + \int_0^t (2(W_1(\omega,s))^2 + 1)e^{(W_1(\omega,s))^2}\, ds \\
&\quad + \int_0^t 2W_1(\omega,s)e^{(W_1(\omega,s))^2}\, dW_1
\end{aligned}$$

(b)

$$\begin{aligned}
Z(\omega,t) &= (W_1(\omega,t) + W_2(\omega,t))^2 \\
&= \left(\int_0^t 1\, dW_1(\omega,s) + \int_0^1 1\, dW_2(\omega,s)\right)^2
\end{aligned}$$

Note that

$$(W_1(\omega,t), W_2(\omega,t)) = \int_0^t b(\omega,s)\, dW$$

where

$$b(\omega,s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Let $g(w_1,w_2) = (w_1+w_2)^2$, so $\bigtriangledown g|_w = (2(w_1+w_2), 2(w_1+w_2))$ and

$$Hf|_w = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\begin{aligned}
&Z(\omega,t) \\
&= \int_0^t \left(\bigtriangledown f|_{(W_1(\omega,s),W_2(\omega,s))} \times 0 + \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \nabla g|_{W_1(\omega,s),W_2(\omega,s)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} dW(\omega,s) \\
= & \int_0^t 2\,ds + \int_0^t (2W_1(\omega,s) + 2W_2(\omega,s))\ dW_1(\omega,s) \\
&+ \int_0^t (2W_1(\omega,s) + 2W_2(\omega,s))\ dW_2(\omega,s)
\end{aligned}
$$

Notice that this can be further manipulated to obtain

$$
\begin{aligned}
&2t + ((W_1(\omega,t))^2 - t) + ((W_2(\omega,t))^2 - t) \\
&\quad +2\int_0^t W_1(\omega,s)\,dW_2(\omega,s) + 2\int_0^t W_2(\omega,s)\,dW_1(\omega,s) \\
= &\ (W_1(\omega,t))^2 + (W_2(\omega,t))^2 + 2\int_0^t W_1(\omega,s)\,dW_2(\omega,s) + 2\int_0^t W_2(\omega,s)\,dW_1(\omega,s)
\end{aligned}
$$

from which it follows that

$$
W_1(\omega,t)W_2(\omega,t) = \int_0^t W_1(\omega,s)dW_2(\omega,s) + \int_0^t W_2(\omega,s)dW_1(\omega,s)
$$

(c) $Z(\omega,t) = e^{W_1(\omega,t)+2W_2(\omega,t)}$ As in part (b),

$$
(W_1(\omega,t), W_2(\omega,t)) = \int_0^t b(\omega,s)\,dW
$$

where

$$
b(\omega,s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Let $g(w_1,w_2) = e^{w_1+2w_2}$, so $\nabla g|_w = (e^{w_1+2w_2}, 2e^{w_1+2w_2})$ and

$$
Hg|_w = \begin{pmatrix} e^{w_1+2w_2} & 2e^{w_1+2w_2} \\ 2e^{w_1+2w_2} & 4e^{w_1+2w_2} \end{pmatrix} = e^{w_1+2w_2} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}
$$

$$
\begin{aligned}
Z(\omega,t) &= 1 + \frac{5}{2}\int_0^t e^{W_1(\omega,s)+2W_2(\omega,s)}\,ds + \int_0^t e^{W_1(\omega,s)+2W_2(\omega,s)} dW_1(\omega,s) \\
&\quad + \int_0^t 2e^{W_1(\omega,s)+2W_2(\omega,s)} dW_2(\omega,s)
\end{aligned}
$$

(d) $Z(\omega,t) = e^{t^2+W_1(\omega,t)}$ We can treat W_1 as a 1-dimensional Wiener process.

$$(t, W_1) = \left(\int_0^t 1\, ds, \int_0^t 1\, dW_1(\omega,s)\right)$$

so

$$a(\omega,t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } b(\omega,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let $g(t,w_1) = e^{t^2+w_1}$, so $\nabla g|_{(t,w_1)} = e^{t^2+w_1}(2t,1)$ and

$$Hg|_{(t,w_1)} = e^{t^2+w_1} \begin{pmatrix} 4t^2+2 & 2t \\ 2t & 1 \end{pmatrix}$$

$$\begin{aligned} Z(\omega,t) &= 1 \\ &+ \int_0^t \left(e^{s^2+W_1(\omega,s)}(2s,1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{e^{s^2+W_1(\omega,s)}}{2}\mathrm{tr}\left((0,1)\begin{pmatrix} 4s^2+2 & 2s \\ 2s & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right) ds \\ &+ \int_0^t e^{s^2+W_1(\omega,s)}(2s,1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_1(\omega,s) \\ &= 1 + \int_0^t \left(e^{s^2+W_1(\omega,s)}\right)\left(2s+\frac{1}{2}\right) ds \\ &+ \int_0^t \left(e^{s^2+W_1(\omega,s)}\right) dW_1(\omega,s) \end{aligned}$$

(e) $Z(\omega,t) = te^{W_1(\omega,t)}$ We can treat W_1 as a 1-dimensional Wiener process. As in part (d),

$$(t, W_1) = \left(\int_0^t 1\, ds, \int_0^t 1\, dW_1(\omega,s)\right)$$

so

$$a(\omega,t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } b(\omega,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let $g(t,w_1) = te^{w_1}$, so $\nabla g|_{(t,w_1)} = (e^{w_1}, te^{w_1})$ and

$$Hg|_{(t,w_1)} = \begin{pmatrix} 0 & e^{w_1} \\ e^{w_1} & te^{w_1} \end{pmatrix} = e^{w_1}\begin{pmatrix} 0 & 1 \\ 1 & t \end{pmatrix}$$

$$
\begin{aligned}
&Z(\omega, t) \\
&= \int_0^t \left(\left(e^{W_1(\omega,s)}, te^{W_1(\omega,s)} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \mathrm{tr} \left((0,1) \left(e^{W_1(\omega,s)} \right) \begin{pmatrix} 0 & 1 \\ 1 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right) ds \\
&\quad + \int_0^t \left(e^{W_1(\omega,s)}, te^{W_1(\omega,s)} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} dW_1(\omega, s) \\
&= \int_0^t \left(e^{W_1(\omega,s)} \right) \left(1 + \frac{s}{2} \right) ds + \int_0^t se^{W_1(\omega,s)} \, dW_1(\omega, s)
\end{aligned}
$$

3. We may assume that $W(\omega, \cdot)$ is continuous for all ω, and hence $Z(\omega, \cdot)$ is continuous for all ω. Suppose that $Z(\omega, t)$ is an Itô process

$$Z(\omega, t) = Z(\omega, 0) + \int_0^t a(\omega, s) \, ds + \int_0^t b(\omega, s) dW(\omega, s)$$

with $a, b \in \mathcal{L}^2$. Then Z and W are measurable processes, i.e. they are measurable on the product $\Omega \times [0, T]$. Let

$$
\begin{aligned}
C &= \{(\omega, t) : W(\omega, t) = Z(\omega, t)\} \\
C_t &= \{\omega : W(\omega, t) = Z(\omega, t)\}
\end{aligned}
$$

By the Law of the Iterated Logarithm in the short run, $P(C_t) = 0$ for all t. C is product measurable on the product since Z and W are measurable processes, so by Fubini's Theorem

$$
\begin{aligned}
(P \otimes \lambda)(C) &= \int_{\Omega \times [0,T]} \mathbf{1}_C(\omega, t) d(P \otimes \lambda) \\
&= \int_0^T \left(\int_\Omega \mathbf{1}_C(\omega, t) \, dP(\omega) \right) d\lambda(t) \\
&= \int_0^T P(C_t) \, d\lambda(t) \\
&= \int_0^T 0 \, d\lambda(t) \\
&= 0
\end{aligned}
$$

If $(\omega, t) \notin C$, then there is a positive ε (depending on ω and t) such that $Z(\omega, s)$ is constant for $s \in (t - \varepsilon, t + \varepsilon)$. This makes it pretty clear that $a(\omega, t) = b(\omega, t)$ almost everywhere on $\omega \times [0, T] \setminus C$.[1] But then

[1] If the Itô integral were defined pathwise, this would be obvious. Giving a formal proof involves arguing through the definition of the Itô integral from simple functions. Alternatively, one can see it immediately from the nonstandard definition of the Itô integral, since the nonstandard definition is given pathwise.

$Z(\omega, t) = Z(\omega, 0) = 0$ almost surely; this contradiction shows that Z is not an Ito process.