

Bus Ad 239B–Spring 2003 Solutions to Problem Set 2

1. This solution is revised, to correct a computational error, and to emphasize a point that several students misunderstood: we need to *construct* \hat{B} fom B , and this needs to be done so that \hat{B} is a *standard* Brownian motion. Because A is symmetric, it has an *orthonormal* basis of eigenvectors, and it can be diagonalized by a unitary transformation constructed from the basis. We need to find the orthonormal basis and unitary transformation in order to construct the *standard* Brownian motion \hat{B} . We diagonalize the matrix

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

The eigenvalues are 5 and 1, with corresponding eigenvectors

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \text{ and } \begin{pmatrix} \Leftrightarrow 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

so if we let

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \sqrt{D} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1/\sqrt{2} & \Leftrightarrow 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

then

$$\begin{aligned} A &= UDU^T \\ &= U\sqrt{D}\sqrt{D}U^T \\ &= U\sqrt{D}(\sqrt{D})^T U^T \\ &= (U\sqrt{D})(U\sqrt{D})^T \\ &= \sigma\sigma^T \end{aligned}$$

where

$$\sigma = (U\sqrt{D}) = \begin{pmatrix} \sqrt{5/2} & \Leftrightarrow\sqrt{1/2} \\ \sqrt{5/2} & \sqrt{1/2} \end{pmatrix}$$

so we must have

$$B = \sigma \hat{B}$$

and therefore

$$\begin{aligned} \hat{B} &= \sigma^{-1} B \\ &= (U\sqrt{D})^{-1} B \\ &= (\sqrt{D})^{-1} U^{-1} B \\ &= (\sqrt{D})^{-1} U^T B \\ &= \begin{pmatrix} 1/\sqrt{5} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1/2} & \sqrt{1/2} \\ \Leftrightarrow 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} B \\ &= \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{10} \\ \Leftrightarrow 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} B \end{aligned}$$

so

$$\hat{B}(\omega, t) = \begin{pmatrix} \frac{B_1(\omega, t) + B_2(\omega, t)}{\sqrt{10}} \\ \frac{-B_1(\omega, t) + B_2(\omega, t)}{\sqrt{2}} \end{pmatrix}$$

We check that \hat{B} is a *standard* Brownian motion. We compute the covariance matrix of \hat{B} from the covariance matrix of B :

$$\begin{aligned} E\left(\left(\hat{B}_1(t)\right)^2\right) &= E\left(\frac{B_1^2(t) + B_2^2(t) + 2B_1(t)B_2(t)}{10}\right) \\ &= \frac{(3 + 3 + 4)t}{10} \\ &= t \\ E\left(\left(\hat{B}_2(t)\right)^2\right) &= E\left(\frac{B_1^2(t) + B_2^2(t) \Leftrightarrow 2B_1(t)B_2(t)}{2}\right) \\ &= \frac{(3 + 3 \Leftrightarrow 4)t}{2} \\ &= t \\ E\left(\hat{B}_1(t)\hat{B}_2(t)\right) &= E\left(\frac{\Leftrightarrow B_1^2(t) + B_2^2(t) + B_1(t)B_2(t) \Leftrightarrow B_1(t)B_2(t)}{20}\right) \\ &= \frac{(\Leftrightarrow 3 + 3)t}{2} \\ &= 0 \end{aligned}$$

This shows that the covariance matrix of \hat{B} is tI , where I is the 2×2 identity matrix. \hat{B} is continuous and has independent increments because B does, so \hat{B} is a standard Brownian motion.

$$\begin{aligned}\sigma\sigma^T &= \begin{pmatrix} \sqrt{5/2} & \Leftrightarrow\sqrt{1/2} \\ \sqrt{5/2} & \sqrt{1/2} \end{pmatrix} \begin{pmatrix} \sqrt{5/2} & \sqrt{5/2} \\ \Leftrightarrow\sqrt{1/2} & \sqrt{1/2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{2} + \frac{1}{2} & \frac{5}{2} \Leftrightarrow\frac{1}{2} \\ \frac{5}{2} \Leftrightarrow\frac{1}{2} & \frac{5}{2} + \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\end{aligned}$$

2. $Z(\omega, t) = e^{\alpha t + \beta B(\omega, t)}$ is a martingale if and only if for all $t \geq s$,

$$\begin{aligned}0 &= E \left(e^{\alpha t + \beta B(\omega, t)} \Leftrightarrow e^{\alpha s + \beta B(\omega, s)} \middle| \mathcal{F}_s \right) \\ &= E \left(\left(e^{\alpha s + \beta B(\omega, s)} \right) \left(e^{\alpha(t-s) + \beta(B(\omega, t) - B(\omega, s))} \Leftrightarrow 1 \right) \middle| \mathcal{F}_s \right) \\ &= \left(e^{\alpha s + \beta B(\omega, s)} \right) E \left(e^{\alpha(t-s) + \beta(B(\omega, t) - B(\omega, s))} \Leftrightarrow 1 \right)\end{aligned}$$

since $B(\cdot, s)$ is \mathcal{F}_s -measurable and $B(\cdot, t) \Leftrightarrow B(\cdot, s)$ is independent of \mathcal{F}_s . Since the first factor is never zero, Z is a martingale if and only if

$$\begin{aligned}E \left(e^{\alpha(t-s) + \beta(B(\omega, t) - B(\omega, s))} \right) &= 1 \\ \Leftrightarrow E \left(e^{\beta(B(\omega, t) - B(\omega, s))} \right) &= e^{-\alpha(t-s)}\end{aligned}$$

Let $X(\omega) = \beta(B(\omega, t) \Leftrightarrow B(\omega, s))$. X is Normal with mean zero and variance $\gamma = \beta^2(t \Leftrightarrow s)$, so

$$\begin{aligned}E \left(e^X \right) &= \frac{1}{\sqrt{2\pi\gamma}} \int_{-\infty}^{\infty} e^r e^{-\frac{r^2}{2\gamma}} dr \\ &= \frac{1}{\sqrt{2\pi\gamma}} \int_{-\infty}^{\infty} e^{\frac{2\gamma r - r^2}{2\gamma}} dr \\ &= \frac{e^{\gamma/2}}{\sqrt{2\pi\gamma}} \int_{-\infty}^{\infty} e^{-\frac{\gamma^2 - 2\gamma r + r^2}{2\gamma}} dr \\ &= \frac{e^{\gamma/2}}{\sqrt{2\pi\gamma}} \int_{-\infty}^{\infty} e^{-\frac{(r-\gamma)^2}{2\gamma}} dr\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{\gamma/2}}{\sqrt{2\pi\gamma}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\gamma}} dr \\
&= e^{\gamma/2}
\end{aligned}$$

so Z is a martingale if and only if

$$\alpha(s \Leftrightarrow t) = \frac{\beta^2(t \Leftrightarrow s)}{2} \Leftrightarrow \alpha = \Leftrightarrow \frac{\beta^2}{2}$$

3.

$$\begin{aligned}
&\int_0^T \hat{X}_n(\omega, t) dX_n(\omega, t) \\
&= \sum_{k=0}^{nT-1} \hat{X}_n\left(\omega, \frac{k}{n}\right) \left(X_n\left(\omega, \frac{k+1}{n}\right) \Leftrightarrow X_n\left(\omega, \frac{k}{n}\right) \right) \\
&= \sum_{k=0}^{nT-1} \sum_{j=1}^k \frac{\omega_j}{\sqrt{n}} \frac{\omega_{k+1}}{\sqrt{n}} \\
&= \sum_{j < k} \frac{\omega_j \omega_k}{n} \\
&= \frac{1}{2} \left(\left(\sum_{j=1}^{nT} \frac{\omega_j}{\sqrt{n}} \right)^2 \Leftrightarrow \sum_{j=1}^{nT} \frac{(\omega_j)^2}{n} \right) \\
&= \frac{1}{2} (X_n(\omega, T)^2 \Leftrightarrow T)
\end{aligned}$$

If you take the integral of X_n rather than \hat{X}_n , the Stieltjes integral you get corresponds to something called the Stratonovich Integral which is studied in stochastic process theory but does not (to my knowledge) have any use in Finance, because X_n is not adapted. Even though X_n and \hat{X}_n are very close together,

$$\sup \left\{ \left| \hat{X}_n(\omega, t) \Leftrightarrow X_n(\omega, t) \right| : \omega \in \Omega, t \in [0, T] \right\} = \frac{1}{\sqrt{n}}$$

$\int \hat{X}_n dX_n$ and $\int X_n dX_n$ are a finite distance apart, something that should not be surprising; if you have inside information, you can make

a finite amount of money by investing very little, and a huge amount of money by investing a modest amount.

$$\begin{aligned}
& \int_0^T X_n(\omega, t) dX_n(\omega, t) \\
&= \sum_{k=0}^{nT-1} \left(X_n \left(\omega, \frac{k}{n} \right) + \frac{\omega_{k+1}}{2\sqrt{n}} \right) \left(X_n \left(\omega, \frac{k+1}{n} \right) \Leftrightarrow X_n \left(\omega, \frac{k}{n} \right) \right) \\
&= \sum_{k=0}^{nT-1} \left(\sum_{j=1}^k \frac{\omega_j}{\sqrt{n}} + \frac{\omega_{k+1}}{2\sqrt{n}} \right) \frac{\omega_{k+1}}{\sqrt{n}} \\
&= \sum_{j < k} \frac{\omega_j \omega_k}{n} + \frac{1}{2} \sum_{j=1}^{nT} \frac{(\omega_j)^2}{n} \\
&= \frac{1}{2} \left(\sum_{j=1}^{nT} \frac{\omega_j}{\sqrt{n}} \right)^2 \\
&= \frac{(X_n(\omega, T))^2}{2}
\end{aligned}$$