Bus Ad 239B–Spring 2003
Solutions to Problem Set 12

The martingale value process is linear, so the value of the coupon bond is just the sum of the values of the coupon payments and the value of the final payment returning the principal. If we assume the coupon bond pays interest continuously at a coupon rate c, and matures at time t, its value at time $s < t$ will be

$$\begin{aligned} P(r(s), t-s, c) &= P(r(s), t-s) + \int_s^t cP(r(s), u-s)\, du \\ &= e^{-m(r(s),t-s)+v(t-s)/2} + \int_s^t ce^{-m(r(s),t-s)+v(u-s)/2}\, du \\ &= e^{-m(r(s),t-s)+v(t-s)/2} + \int_s^t ce^{-m(r(s),t-s)+v(u-s)/2}\, du \end{aligned}$$

where we recall that

$$m(r,\tau) = \tau\bar{r} + \frac{1}{a}\left(1 - e^{-a\tau}\right)(r - \bar{r}), \quad v(\tau) = \frac{\sigma^2}{2a^3}\left(re^{-a\tau} - e^{-2a\tau} + 2a\tau - 3\right)$$

and a, σ and $\bar{r}$ are the parameters of the Ornstein-Uhlenbeck Process driving the spot interest rate (in particular, $\bar{r}$ is the mean-reversion target with respect to the risk-adjusted measure Q). You weren't asked to determine the differential of the bond price, but notice it is possible to do it in essentially the same way as in Section 4.2 of Nielsen. Since the total return on the bond is the sum of the coupon payments and the drift of the price of the coupon bond, the relative drift of the coupon bond under the risk-adjusted probabilities at time u must equal

$$r(u) - \frac{c}{P(r(u), t-u, c)}$$

In particular, if c is sufficiently large, we can have $P(r(u), t-u, c) > 1$ and a negative relative drift. The dispersion of P will be an integral of the dispersions of the coupons, added to the dispersion of the final payment.

The yield of the bond is defined as the interest rate R such that the value of the bond equals

$$PDV(R, s) = e^{-(t-s)R} + \int_s^t ce^{-(u-s)R}\, du$$

the present discounted value of the coupon and final payments. Notice that PDV is differentiable and strictly decreasing in R, $PDV \to \infty$ as $R \to -\infty$, and $PDV \to 0$ as $R \to \infty$, so there is a unique $R(s)$ such that $P(r(s), t - s, c) = R(s)$. Even though we don't have a closed form expression for $R(s)$, we can determine its differential:

$$\begin{aligned}
P(r(s), t-s, c) &= e^{-(t-s)R} + \int_s^t ce^{-(u-s)R}\, du \\
dP &= \left(-(t-s)e^{-(t-s)R} - \int_s^t (u-s)ce^{-(u-s)R}\, du\right) dR \\
dR &= \frac{dP}{\left(-(t-s)e^{-(t-s)R} - \int_s^t (u-s)ce^{-(u-s)R}\, du\right)}
\end{aligned}$$