Bus Ad 239B–Spring 2003
Solutions to Problem Set 11

1. Recall $S(t) = e^{\gamma t + \sigma W^\lambda}$, where $\gamma = r - \frac{\sigma^2}{2}$ and W^λ is a standard Wiener process with respect to the risk-adjusted probability measure Q. Let N be the cumulative distribution function of the standard normal. The martingale value process of the put option is given by

$$
\begin{aligned}
\Phi(S,t) &= e^{-r(T-t)} E_Q(g(S(T))|S(t) = S) \\
&= e^{-r(T-t)} E_Q(\max\{X - S(T), 0\}|S(t) = S) \\
&= e^{-r(T-t)} \int_{-\infty}^{\infty} \max\{0, X - Se^{\gamma(T-t)+\sigma\sqrt{T-t}y}\}\, dN(y) \\
&= e^{-r(T-t)} \int_{-\infty}^{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}} X\, dN(y) \\
&\qquad - e^{-r(T-t)} \int_{-\infty}^{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}} Se^{\gamma(T-t)+\sigma\sqrt{T-t}y}\, dN(y) \\
&= Xe^{-r(T-t)} \int_{-\infty}^{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}} dN(y) \\
&\qquad - Se^{(\gamma-r)(T-t)} \int_{-\infty}^{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}} e^{\sigma\sqrt{T-t}y}\, dN(y) \\
&= Xe^{-r(T-t)} N\left(\frac{\ln(X/S) - \gamma(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\qquad - Se^{(\gamma - r + \sigma^2/2)(T-t)} N\left(\frac{\ln(X/S) - \gamma(T-t)}{\sigma\sqrt{T-t}} - \sigma\sqrt{T-t}\right) \\
&= Xe^{-r(T-t)}\left(1 - N\left(\frac{\ln(S/X) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&\qquad - S\left(1 - N\left(\frac{\ln(S/X) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&= e^{-r(T-t)} X(1 - N(d_2)) - S(1 - N(d_1))
\end{aligned}
$$

Therefore, if P is the put and C is the call,

$C(t) - P(t)$

$$
\begin{aligned}
&= N(d_1)S - e^{-r(T-t)}XN(d_2) + (1-N(d_1))S - e^{-r(T-t)}X(1-N(d_2)) \\
&= S - e^{-r(T-t)}X
\end{aligned}
$$

which establishes put-call parity.

2. Recall $S(t) = e^{\gamma t+\sigma W}$, where W is a standard Wiener process and $\gamma = \mu - \sigma^2/2$. Let $\lambda = \frac{\mu-r}{\sigma}$. Since λ is a constant, $\Pi = \eta[-r,-\lambda]$ is a state price process. The martingale value process is given by

$$
\begin{aligned}
\Phi(S,t) &= \frac{1}{\Pi(t)}E(\Pi(T)g(S(T))|S(t)=S) \\
&= \frac{1}{e^{(-r-\lambda^2/2)t-\lambda W(t)}}E\left(e^{(-r-\lambda^2/2)T-\lambda W(T)}\max\{S(T)-X,0\}\Big|\, S(t)=S\right) \\
&= e^{(-r-\lambda^2/2)(T-t)}E\left(e^{-\lambda(W(T)-W(t))}\max\{S(T)-X,0\}\Big|\, S(t)=S\right) \\
&= e^{(-r-\lambda^2/2)(T-t)}\int_{-\infty}^{\infty} e^{-\lambda\sqrt{T-t}y}\max\{0, Se^{\gamma(T-t)+\sigma\sqrt{T-t}y}-X\}\,dN(y) \\
&= e^{(-r-\lambda^2/2)(T-t)}\int_{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}}^{\infty} Se^{\gamma(T-t)+(\sigma-\lambda)\sqrt{T-t}y}\,dN(y) \\
&\qquad -e^{(-r-\lambda^2/2)(T-t)}\int_{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}}^{\infty} e^{-\lambda\sqrt{T-t}y}X\,dN(y) \\
&= Se^{(-r-\lambda^2/2+\gamma)(T-t)}\int_{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}}^{\infty} e^{(\sigma-\lambda)\sqrt{T-t}y}\,dN(y) \\
&\qquad -Xe^{(-r-\lambda^2/2)(T-t)}\int_{\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}}^{\infty} e^{-\lambda\sqrt{T-t}y}\,dN(y) \\
&= Se^{(-r-\lambda^2/2+\gamma+(\sigma-\lambda)^2/2)(T-t)}\left(1-N\left(\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}-(\sigma-\lambda)\sqrt{T-t}\right)\right) \\
&\qquad -Xe^{(-r-\lambda^2/2+\lambda^2/2)(T-t)}\left(1-N\left(\frac{\ln(X/S)-\gamma(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&= Se^{(-r+\gamma+\sigma^2/2-\sigma\lambda)(T-t)}N\left(\frac{\ln(S/X)+\left(\mu-\frac{\sigma^2}{2}+\sigma^2-\sigma\lambda\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\qquad -Xe^{-r(T-t)}N\left(\frac{\ln(S/X)+\left(\mu-\frac{\sigma^2}{2}-\sigma\lambda\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= Se^{(-r+\mu-\sigma^2/2+\sigma^2/2-(\mu-r))(T-t)}N\left(\frac{\ln(S/X)+\left(\mu+\frac{\sigma^2}{2}-(\mu-r)\right)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \quad -Xe^{-r(T-t)}N\left(\frac{\ln(S/X)+\left(\mu-\frac{\sigma^2}{2}-(\mu-r)\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= SN\left(\frac{\ln(S/X)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
& \quad -Xe^{-r(T-t)}N\left(\frac{\ln(S/X)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= SN(d_1)-e^{-r(T-t)}XN(d_2)
\end{aligned}
$$