Bus Ad 239B–Spring 2003
Solutions to Problem Set 10

1. (a) Let $\bar{S}$ be an Itô process. $\Pi(\omega, t) = 1$ is a state price process for $\bar{S}$ if and only if $\Pi\bar{S} = \bar{S}$ has zero drift. Thus, the set of $\bar{S}$ for which Π is a state price process is

$$\{\bar{S} : \bar{S}(\omega, t) = \bar{S}(\omega, 0) + \int_0^t \bar{\sigma}\, dW \text{ for some } \bar{\sigma} \in \mathcal{L}^2\}$$

Note that by the Martingale Representation Theorem, this includes all $\bar{S}$ in which each component $\bar{S}_n$ is a martingale on the filtration generated by the Wiener process.

(b) Let

$$\bar{S}(t) = \bar{S}(0) + \int_0^t \bar{\mu}\, ds + \int_0^t \bar{\sigma}\, dW$$

$\Pi = \eta[-r, 0]$, so Π is a state price process if and only if

$$\bar{\mu} - r\bar{S} = \bar{\sigma}\lambda^T = 0$$

so $\bar{\mu} = r\bar{S}$ and the set of possible security price processes is the set of all processes $\bar{S}$ satisfying

$$\bar{S}(t) = \bar{S}(0) + \int_0^t r\bar{S}\, ds + \int_0^t \bar{\sigma}\, dW$$

for some $\bar{\sigma} \in \mathcal{L}^2$. Notice that every security has the same instantaneous interest rate r, regardless of the level of risk or the corrleation of the risk with the risk on the other securities.

(c) Let

$$\bar{S}(t) = \bar{S}(0) + \int_0^t \bar{\mu}\, ds + \int_0^t \bar{\sigma}\, dW$$

$\Pi = \eta[0, -\lambda]$, so let $r = 0$; then Π is a state price process if and only if

$$\bar{\mu} - r\bar{S} = \bar{\sigma}\lambda^T$$

so $\bar{\mu} = \bar{\sigma}\lambda^T$ and the set of possible security price processes is the set of all processes $\bar{S}$ satisfying

$$\bar{S}(t) = \bar{S}(0) + \int_0^t (\bar{\sigma}\lambda^T)\, ds + \int_0^t \bar{\sigma}\, dW$$

for some $\bar{\sigma} \in \mathcal{L}^2$.

2. If Π is a state price process for

$$\bar{S}(t) = \bar{S}(0) + \int_0^t \bar{\mu}\, ds + \int_0^t \bar{\sigma}\, dW$$

then we must have

$$\bar{\mu} - r\bar{S} = \bar{\sigma}\lambda^T \tag{1}$$

When $N > K$, this linear equation will typically have no solution. If we consider a fixed constant $\bar{\sigma}$, then for each (ω, t)

$$\{\bar{\mu} \in \mathbf{R}^{N+1} : \text{ there is a solution to (1)}\}$$

is a subspace of dimension at most $1 + \operatorname{rank} \bar{\sigma} \leq K + 1$ in $\mathbf{R}^{N+1}$, (the 1 comes from the ability to set the parameter r) which is a set of Lebesgue measure zero.

To get a cleaner theorem, we consider securities prices $\bar{S}$ satisfying the stochastic differential equation

$$\frac{d\bar{S}}{\bar{S}} = \bar{\mu}\, dt + \bar{\sigma}\, dW$$

where

$$\frac{d\bar{S}}{\bar{S}} = \begin{pmatrix} \frac{d\bar{S}_0}{\bar{S}_0} \\ \vdots \\ \frac{d\bar{S}_N}{\bar{S}_N} \end{pmatrix}$$

and $\bar{\mu}$ and $\bar{\sigma}$ are constants. As we saw in Problem Set 7, the condition for $\eta[-r, -\lambda]$ to be a state price process when the stochastic differential equation is in this form is

$$\bar{\mu} - \begin{pmatrix} r \\ \vdots \\ r \end{pmatrix} = \bar{\sigma}\lambda^T \tag{2}$$

For fixed $\bar{\sigma}$, the set of $\bar{\mu}$ for which Equation (2) has a solution is a set of dimension at most $K + 1$ in $\mathbf{R}^{N+1}$, and hence is a set of measure zero. Alternatively, this implies that the set of pairs $(\bar{\sigma}, \bar{\mu})$ for which Equation (2) is a set of Lebesgue measure zero.

Contrary to what you might think, this is not a negative result. It says that the existence of a state price process imposes some discipline on securities prices when there are more risky securities than components of the Wiener process. The dispersion of the risky securities must exhibit linear dependence; the instantaneous means must exhibit a matching linear dependence; if not, one can construct two instantaneously riskless securities with different mean returns, and hence an arbitrage will exist.