Bus Ad 239B–Spring 2003
Solutions to Problem Set 1

1. Let $X_{n\sigma}$ be the random walk with steps of the form $\frac{\pm\sigma}{\sqrt{n}}$, and let B_σ be the limit process as $n \to \infty$. As in class, $X_{n\sigma}(\cdot, t) - X_{n\sigma}(\cdot, s)$ will be approximately normal with mean zero and variance $(t-s)\sigma^2$ by the Central Limit Theorem. The properties of the limit ($B_\sigma(\omega, 0) = 0$, continuity of paths, $B_\sigma(\cdot, t) - B_\sigma(\cdot, s)$ is normal, mean zero, variance $(t-s)\sigma^2$, independent increments) follow in the same way as in class. If we let $Z(\omega, t) = \frac{B_\sigma(\omega,t)}{\sigma}$, then $Z(\omega, t)$ satisfies all the properties of a one-dimensional standard Brownian motion; in other words, $B_\sigma = \sigma Z$, where Z is a standard Brownian motion. To see that B_σ is a standard Brownian motion with a time change, define $\hat{B}(\omega, t) = B_\sigma(\omega, t/\sigma^2)$. Then $\hat{B}$ is a standard Brownian motion and $B_\sigma(\omega, t) = \hat{B}(\omega, \sigma^2 t)$.

2. Let $\hat{B}(\omega, t) = tB(\omega, 1/t)$ for $t > 0$, $\hat{B}(\omega, 0) = 0$. By Proposition 1.6, $\hat{B}$ is a standard Brownian motion. Then using the substitution $s = 1/t$,

$$\begin{aligned}
&\limsup_{t\to 0} \frac{B(\cdot, t)}{\sqrt{2t \ln|\ln t|}} \\
&= \limsup_{s\to\infty} \frac{B(\cdot, 1/s)}{\sqrt{\frac{2\ln|-\ln s|}{s}}} \\
&= \limsup_{s\to\infty} \frac{sB(\cdot, 1/s)}{\sqrt{2s \ln\ln s}} \\
&= \limsup_{s\to\infty} \frac{\hat{B}(\cdot, s)}{\sqrt{2s \ln\ln s}} \\
&= 1
\end{aligned}$$

almost surely by the Iterated Logarithm Law in the Long Run.

3. There is a slight error in the statement; the first sum in the question should run from $k = 0$ to $nT - 1$, rather than from $k = 1$ to nT, and analogous changes in the parts; my apologies for this.

(a) Between $t_k = \frac{2k}{n}$ and $t_{k+1} = \frac{2k+2}{n}$, there are two coin tosses. Thus,

$$X(\omega, t_{k+1}) - X(\omega, t_k) = \begin{cases} \frac{2}{\sqrt{n}} & \text{with probability } \frac{1}{4} \\ 0 & \text{with probability } \frac{1}{2} \\ -\frac{2}{\sqrt{n}} & \text{with probability } \frac{1}{4} \end{cases}$$

so

$$\left(X(\omega, t_{k+1}) - X(\omega, t_k)\right)^2 = \begin{cases} \frac{4}{n} & \text{with probability } \frac{1}{2} \\ 0 & \text{with probability } \frac{1}{2} \end{cases}$$

Since $X(\omega, t_{k+1}) - X(\omega, t_k)$ is independent of $X(\omega, t_{j+1}) - X(\omega, t_j)$ whenever $j \neq k$,

$$\begin{aligned} E\left(\sum_{k=0}^{\lfloor \frac{nT}{2} \rfloor - 1} \left(X(\omega, t_{k+1}) - X(\omega, t_k)\right)^2\right) &= \lfloor \frac{nT}{2} \rfloor \frac{2}{n} \to T \\ Var\left(\sum_{k=0}^{\lfloor \frac{nT}{2} \rfloor - 1} \left(X(\omega, t_{k+1}) - X(\omega, t_k)\right)^2\right) &= \lfloor \frac{nT}{2} \rfloor \frac{4}{n^2} \to 0 \end{aligned}$$

so

$$\sum_{k=0}^{\lfloor \frac{nT}{2} \rfloor - 1} \left(X(\omega, t_{k+1}) - X(\omega, t_k)\right)^2 \to T$$

in probability.

(b) We construct the partition $\{t_k(\omega) : 0 \leq k \leq m(n, \omega)\}$ as follows: for all ω, all of the times $\frac{2j}{n}$ are included; in addition, we include each time of the form $\frac{2j+1}{n}$ if and only if $\omega_{2j+2} = -\omega_{2j+1}$. In other words, between each pair of successive even times $\frac{2j}{n}$ and $\frac{2j+2}{n}$, we include the odd time $\frac{2j+1}{n}$ in the partition for ω if the two steps in the random walk go in opposite directions. The contribution to the quadratic variation between times $\frac{2j}{n}$ and $\frac{2j+2}{n}$, using this partition, is $\frac{4}{n}$ with probability $\frac{1}{2}$ (when the two steps go in the same direction) and $\frac{2}{n}$ with probability $\frac{1}{2}$ (when the two steps go in the opposite direction), so the expected contribution to the quadratic variation is $\frac{3}{n}$; as in part (a), this implies that the quadratic variation for this sequence of partitions converges in probabilty to $\frac{3T}{2}$.

(c) The random walk paths are piecewise linear; if one takes the partition finer than the random walk time set, the quadratic variation with respect to the partition can be made less than T (to see this, compute for yourself the quadratic variation if one takes $t_k = \frac{k}{2n}$ ($k = 0, \ldots, 2nT$)). So we need to assume that $t_k(\omega) \in \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, T\right\}$). If, in addition, $t_k(\omega)$ is independent of ω and $\max\{t_{k+1} - t_k\} \to 0$, it is not hard to modify the argument in part (a) to show that the quadratic variation converges to T, but this is a stronger assumption than we need. The thing that drives the example in part (b) is that the decision whether to include the odd time node $\frac{2j+1}{n}$ depends on the next step ω_{2j+2} of the random walk. In other words, the decision on whether to include $\frac{2j+1}{n}$ in the partition depends on information that is not known at $\frac{2j+1}{n}$, and this is what drives the example. We can rule this out by requiring that each $t_k(\omega)$ be a *stopping time*: $\{\omega : t_k(\omega) \leq t\} \in \mathcal{F}_t$. The quadratic variation converges to T provided that

- $t_k(\omega) \in \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, T\right\}$
- $\max\{\max\{t_{k+1}(\omega) - t_k(\omega)\} : \omega \in \Omega\} \to 0$
- each t_k is a stopping time.

4. The trick here is an observation called the Reflection Principle. Suppose that nT is odd.[1] Pick $x = \ell/\sqrt{n}$ with ℓ an even natural number, so x is a possible value for the maximum of the random walk. Let ω be a path such that $\max_{t \in [0,T]} X(\omega, t) \geq x$. Let $k = n \min\{t : X(\omega, t) = x\}$; note that k must be even. Now define ω' by

$$\omega'_j = \begin{cases} \omega_j & \text{if} \quad j \leq k \\ -\omega_j & \text{if} \quad j > k \end{cases}$$

Since $nT - k$ is odd, we cannot have $X(\Omega, T) = x$. Thus, exactly one of two cases must occur:

- $X(\omega, T) < x$ and $X(\omega', T) > x$; or
- $X(\omega, T) > x$ and $X(\omega', T) < x$.

[1] If nT is even, take x of the form $\ell/\sqrt{n}$ with ℓ odd, and the same conclusion follows.

In other words, the probability that $\max_{t\in[0,T]} X(\omega,t) \geq x$ is exactly twice the probability that $X(\omega,T) > x$. Since standard Brownian motion B is the limit of the random walk X as $n \to \infty$,

$$\begin{aligned} & P(\{\omega : \max_{t\in[0,T]} B(\omega,t) > y\}) \\ &= 2P(\{\omega : B(\omega,T) > y\}) \\ &= \sqrt{\frac{2}{\pi}} \int_{y/T}^{\infty} e^{-z^2/2} dz \end{aligned}$$

for $y \geq 0$.