Bus Ad 239B–Spring 2003
Problem Set 9
Due Thursday April 3

Find the flaw(s) in the following argument:

1. Consider a variant of the standard Black-Scholes Model. There is one stock S satisfying
$$\frac{dS}{S} = \mu\, dt + \sigma\, dW$$
The argument can be modified to work for general constants μ and σ, but for simplicity, we assume $\mu = 0$. There is also one money-market account M with constant interest rate r. Again, for simplicity, we assume that $r = 0$, so $M = 1$. Let
$$\bar{S} = \begin{pmatrix} M \\ S \end{pmatrix}$$
Thus,
$$d\bar{S} = \bar{\mu}\, dt + \bar{\sigma}\, dW$$
where
$$\bar{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \bar{\sigma} = \begin{pmatrix} 0 \\ S\sigma \end{pmatrix}$$

2. Consider a standard call option with exercise price X at date T, and suppose that $S(0) \leq X$.

3. Let
$$\bar{\Delta}(\omega, t) = \begin{cases} (-X, 1) & \text{if} \quad S(\omega, t) \geq X \\ (0, 0) & \text{if} \quad S(\omega, t) < X \end{cases}$$
In other words, $\bar{\Delta}$ prescribes zero holdings if the stock price is below the strike price; as the stock price hits the strike price, $\bar{\Delta}$ prescribes borrowing X in the money market and using it to buy one unit of the stock, then holding that position unless and until the stock price hits X again.

4. $\bar{\Delta}$ is measurable and adapted. $\bar{\Delta}$ is a buy-and-hold strategy except when $S(\omega, t) = X$, and it is self-financing at the nodes where $S(\omega, t) = X$, so it is self-financing.

5. $\bar{\Delta}(0)\bar{S}(0) = 0$ and

$$\bar{\Delta}(T)\bar{S}(T) = \begin{cases} S(T) - X & \text{if} \quad S(T) \geq X \\ 0 & \text{if} \quad S(T) < X \end{cases}$$

   In particular, $\bar{\Delta}$ replicates the call option.

6. $\bar{\Delta}\bar{\mu} = 0$ and $\bar{\Delta}\bar{\sigma} = S\sigma \in \mathcal{H}^2$, so $\bar{\Delta}\bar{S}$ is a martingale.

7. Since $\bar{S}$ has zero drift, $\Pi = 1$ is a state price process. $\Pi\bar{\Delta}\bar{S}$ is a martingale, so $\bar{\Delta}$ is admissible with respect to $\bar{S}$ and Π.

8. $E(\bar{\Delta}(T)\bar{S}(T)) > 0$ and $\bar{\Delta}(T)\bar{S}(T) \geq 0$, but $\bar{\Delta}(0)\bar{S}(0) = 0$, so $\bar{\Delta}$ is a self-financing arbitrage strategy.