Bus Ad 239B–Spring 2003
Problem Set 5
Due Thursday February 27

1. Let W be a standard 1-dimensional Wiener process. Answer questions (a)-(e) for each of the following two processes:

$$Z(\omega, t) = W(\omega, t)^2 \text{ and } Z(\omega, t) = e^{(\mu-2)t+2W(\omega,t)}$$

   (a) Write $Z = M + A$, where M is a martingale and A is of bounded variation almost surely over each bounded time interval.

   (b) Construct a standard Wiener process $\hat{M}$ such that M is a time change of $\hat{M}$.

   (c) Is there an equivalent probability measure Q such that Z is a martingale with respect to Q? If so, construct it.

   (d) Is there an equivalent probability measure Q such that M is a standard Wiener process with respect to Q?

2. Consider the random walk version of the Brownian bridge discussed in class, i.e Fix n even, T such that $nT \in \mathbf{N}$, let X be the standard binary random walk described in Lecture 1, and consider the conditioned process

$$B(\omega, t) = X(\omega, t | X(\omega, T) = 0)$$

As noted in class, if we define $\Omega' = \{\omega \in \Omega : X(\omega, T) = 0\}$ with the probability measure $P(A) = \frac{|A|}{|\Omega'|}$, then we can think of B as the restriction of X to Ω'. Compute $\sigma^2(\omega, k, n)$, the variance of $B\left(\omega, \frac{k+1}{n}\right) - B\left(\omega, \frac{k}{n}\right)$. Now suppose $n \to \infty$ and $\frac{k_n}{n} \to t \in [0, T)$. Show that

$$n\sigma^2(\omega, k_n, n) \to 1$$

(you will need to decide what sense of convergence to use). What happens if $k_n/n \to T$? Use these facts to derive (you can be informal) the stochastic differential equation satisfied by the Brownian bridge

$$W(\omega, t | W(\omega, T) = 0)$$

where W is a standard Wiener process.