Bus Ad 239B–Spring 2003
Problem Set 1
Due Thursday January 30

1. Consider a one-dimensional random walk process like the random walk process X discussed in class, except that every step is of the form $\frac{\pm\sigma}{\sqrt{n}}$ rather than $\frac{\pm 1}{\sqrt{n}}$. What is the limit of this random walk as $n \to \infty$ (you may argue informally, as in class)? Show that the limit is a standard one-dimensional Brownian motion with a time change.

2. Let B be a standard one-dimensional Brownian motion. Evaluate
$$\limsup_{t\to 0} \frac{B(\cdot, t)}{\sqrt{2t \ln |\ln t|}}$$
Hint: Use Proposition 1.6.

3. Let X be the one-dimensional random walk discussed in class. We noted in class that, with respect to the partition $t_k = \frac{k}{n}$, the quadratic variation
$$\sum_{k=0}^{nT-1} \left(X(\omega, t_{k+1}) - X(\omega, t_k)\right)^2 = T$$
for all ω.

   (a) Show that if $t_k = \frac{2k}{n}$ for $k = 1, \ldots, \lfloor \frac{nT}{2} \rfloor$, then
$$\sum_{k=0}^{\lfloor \frac{nT}{2} \rfloor - 1} \left(X(\omega, t_{k+1}) - X(\omega, t_k)\right)^2 \to T$$
   in probability as $n \to \infty$.

(b) Show that, if one is allowed to choose t_k as a function of ω ($k = 1, \ldots, m(n, \omega)$), then one can find a choice of $t_k(\omega)$ such that

$$\sum_{k=0}^{m(n,\omega)} \left(X(\omega, t_{k+1}(\omega)) - X(\omega, t_k(\omega))\right)^2 \to \frac{3T}{2}$$

in probability as $n \to \infty$.

(c) Conjecture what assumption is needed on the $t_k(\omega)$ to ensure that

$$\sum_{k=0}^{m(n,\omega)} \left(X(\omega, t_{k+1}(\omega)) - X(\omega, t_k(\omega))\right)^2 \to T$$

in probability as $n \to \infty$.

4. Let X be the one-dimensional random walk discussed in class. Find the distribution of the random variable

$$\max_{t \in [0,T]} X(\cdot, t)$$

and deduce the distribution of the random variable

$$\max_{t \in [0,T]} B(\cdot, t)$$

where B is standard one-dimensional Brownian motion. Hint: Find a relationship between $\max_{t \in [0,T]} X(\cdot, t)$ and $X(\cdot, T)$.