

## COMMENT

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Professor Uzawa's excellent paper brings the powerful analytic tools of neo-classical growth theory to bear on the question of aggregate fiscal policy for optimal development. His conclusions are potentially of great use in examining the broad issues of aggregate planning. Perhaps more important, the paper marks the beginning of a convergence of growth theory and development theory. This merger will result in the addition of the mathematical techniques of growth theory to the development economist's bag of tools and an increase in the practicality and applicability of growth models.

Before discussing Professor Uzawa's conclusions and their implications, it will be useful to review the model he has used. I will give a free translation of his model which I think has some advantages in simplicity and added generality.

## I. VARIABLES

In his analysis, Professor Uzawa has worked with variables expressed in "per unit of capital stock" terms. However, the optimal growth portions of the paper are handled more naturally with "per man" variables, and empirical data are more commonly collected in "per man" terms. For these reasons, I will choose the "per man" alternative. I will put primes on variables to indicate that they are now in "per man in the population" rather than "per unit of capital" terms.

- $k$  = aggregate capital per man (in population).
- $y'_p$  = output of private goods per man.
- $y'_g$  = output of public goods per man.
- $p$  = imputed price of public goods, measured in units of private goods.
- $y'$  = real national income per man (private goods are taken as numéraire, so that all real quantities are measured in units of private goods).
- $y^d$  = real disposable income per man.
- $\rho$  = money interest rate.
- $m'$  = real cash balances per man.
- $c'$  = real consumption of private goods per man.
- $z'$  = real investment of private goods per man.
- $a'$  = real value of assets per man.
- $\theta$  = rate of increase of real money supply (money supply = net liabilities of public sector to private sector).
- $\tau$  = income tax rate on national income.

We have a total of thirteen variables, among which  $k$  is predetermined,  $\theta$  and  $\tau$  are instrument variables, and the remaining ten are endogenous.

## II. RELATIONS

Professor Uzawa takes the wage-rental ratio as his "key" variable in analyzing the model and works explicitly with the production functions and factor markets. We take an alternative approach in which the output of public goods per man is our "key" variable, the production frontier is explicit, and all factor markets are implicit. This alternative has the advantage of not requiring the "no factor intensity reversal" condition assumed by Professor Uzawa.

Two goods are produced in the model: public goods which are consumed and private goods which are either consumed or accumulated as capital stock (capital can then be measured in units of private goods).

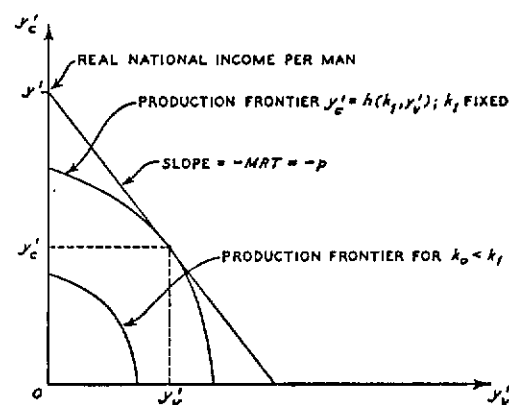


Figure 1. Production Frontier

The relation between capital per man and per capita outputs of public and private goods is given by a production frontier illustrated in Figure 1 and having a formula:

$$(1) \quad y'_g = h(k, y'_p).$$

Under the assumptions Professor Uzawa has placed on the production functions, the frontier  $h$  is concave in  $(k, y'_p)$ , strictly increasing in  $k$ , and strictly decreasing in  $y'_p$ . Further, the function is *strictly* concave except for  $k$  values at which a factor intensity reversal occurs, resulting in  $y'_g$  depending linearly on  $y'_p$ .

The imputed price of public goods is given by the marginal rate of transformation of the production frontier:

$$(2) \quad p = -h_2(k, y'_p) \quad (\text{Notation: } h_2 = \partial h / \partial y'_p).$$

From (1) and (2), we can now define real national income:

$$(3) \quad y' = y'_e + py'_e = h(k, y'_e) - y'_e h_2(k, y'_e).$$

Equation (3) defines  $y'$  as a function  $y' = g(k, y'_e)$ . From Figure 1,  $g$  is non-decreasing in  $y'_e$  and increasing except at a factor reversal. The sign of  $\partial g / \partial k$  is not generally determinate. However, if the private sector is more capital-intensive than the public sector, then  $g$  is strictly increasing in  $k$ .

The balance-of-the-budget equation for the public sector is

$$(4) \quad \theta m' = py'_e - \tau y' = (1 - \tau)y' - y'_e,$$

and states that the public sector meets a deficit in its budget by increasing the real money supply.

Balance in the private sector requires

$$(5) \quad y'_e = c' + z'.$$

Real assets per man equal capital stock held per man plus real cash balances held per man,

$$(6) \quad a' = k + m'.$$

Disposable income is related to national income by

$$(7) \quad y^{d'} = (1 - \tau)y'.$$

Consumption of private goods per man is given by a consumption function,

$$(8) \quad c' = C(\rho, y^{d'}, a'),$$

where  $0 \geq C_1 [= \partial C / \partial \rho]$ ;  $0 < C_2$ ;  $0 \leq C_3$ ; and  $C$  is concave in  $y^{d'}$  and  $a'$  with  $C(\rho, 0, 0) = 0$ .<sup>1</sup>

<sup>1</sup> Professor Uzawa assumes further that  $C$  is linear homogeneous in the last two variables. Our assumption implies nonincreasing average propensities to consume.

Investment per man is given by an investment function,

$$(9) \quad z' = Z'(\rho, y', k)$$

where  $0 > Z'_1 [= \partial Z' / \partial \rho]$ ;  $0 \leq Z'_2$ , and the elasticity of  $z'$  with respect to  $k$  is no greater than one ( $\partial \log Z' / \partial \log k \leq 1$ ). This class of investment functions includes the one used by Professor Uzawa, which in terms of our variables takes the form  $z' = kZ(\rho, y'/k)$ .

The portfolio balance of assets is determined by a liquidity preference function,

$$(10) \quad m' = \lambda'(\rho, y^{d'}, k)$$

where  $0 > \lambda'_1 [= \partial \lambda' / \partial \rho]$ , and the elasticities of  $m'$  with respect to  $y^{d'}$  and  $k$  lie between zero and one ( $0 \leq \partial \log \lambda' / \partial \log y^{d'} \leq 1$  and  $0 \leq \partial \log \lambda' / \partial \log k \leq 1$ ). This class of liquidity preference functions includes the one used by Professor Uzawa, which in terms of our variables takes the form  $m' = k\lambda(\rho, y^{d'}/k)$ .

### III. ANALYSIS

The system (1)–(10) of ten equations in  $k, \tau, \theta$ , and the ten endogenous variables can be simplified considerably by elimination of variables. Using equation (1), substitute  $h$  for  $y_e$  in the remaining equations. Substituting (2) into (3) yields

$$(11) \quad y' = g(k, y'_e) \equiv h(k, y'_e) - y'_e h_2(k, y'_e).$$

Using (7), eliminate  $y^{d'}$  from the rest of the equations. Then, using (10), eliminate  $m'$  from equations (4) and (6). The public sector balance equation (4) becomes

$$(12) \quad \theta \lambda'(\rho, (1 - \tau)y', k) = (1 - \tau)y' - h(k, y'_e).$$

Substituting (6) into (8), and then (8) and (9) into (5) yields the private sector balance equation:

$$(13) \quad h(k, y'_e) - C'(\rho, (1 - \tau)y', k + \lambda'(\rho, (1 - \tau)y', k)) = Z'(\rho, y', k).$$

Equations (11)–(13) give three equations in the three endogenous variables,  $\rho, y'$ , and  $y'_e$ . Substituting  $y' = g(k, y'_e)$  in (11) into (12), one can solve for  $\rho$  as a function  $\rho = \rho'(y'_e; k, \theta, \tau)$  of  $y'_e$  and the variables  $(k, \theta, \tau)$ . Substituting this solution into (12) and differentiating,

$$(14) \quad \theta \lambda'_1 \frac{\partial \rho'}{\partial y'_e} = [(1 - \tau) - (1 - \tau)\theta \lambda'_2] \frac{\partial g}{\partial y'_e} - h_2.$$

Now, the term in square brackets in (14) is nonnegative:  $0 \leq \partial \log \lambda' / \partial \log y^{d'} \leq 1$  implies  $\lambda'_2 \leq \lambda' / y^{d'}$ , giving  $[1 - \theta \lambda'_2] \geq [1 - \theta \lambda' / y^{d'}] = h / (1 - \tau)y'$  by (12). Since  $h_2 < 0$ ,  $\theta \lambda'_1 \partial \rho' / \partial y'_e$  is positive, implying  $\partial \rho' / \partial y'_e$  negative. Further differentiation shows that  $\rho'$  is increasing in  $\theta$  and increasing in  $\tau$ . In general,  $\partial \rho' / \partial k$  is not determinate in sign. However, if the private sector is less capital-intensive than the public sector, then  $\rho$  falls when  $k$  rises, implying  $\partial g / \partial k \leq \partial h / \partial k$  and  $\partial \rho' / \partial k$  positive. These conclusions are illustrated in Figure 2.

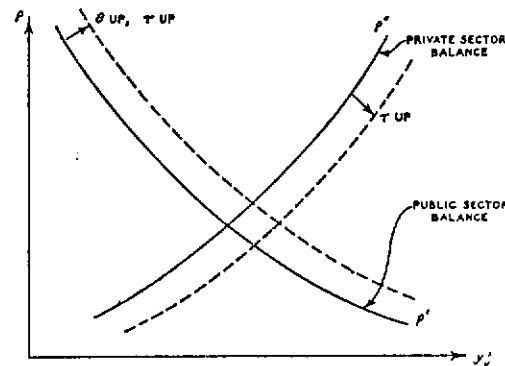


Figure 2

Next substitute  $y' = g(k, y'_s)$  in (11) into (13). The left- and right-hand sides of (13) are plotted in Figure 3. The right-hand side is a strictly decreasing function of  $y'_s$ , is strictly increasing in  $\rho$  and  $\tau$ , and is independent of  $\theta$ . The left-hand side is strictly decreasing in  $\rho$ , nondecreasing in  $y'_s$ , and independent of  $\theta$  and  $\tau$ .

The solution of (13) in Figure 3 defines  $\rho$  as a function  $\rho = \rho''(y'_s; k, \tau)$  which is strictly increasing in  $y'_s$  and strictly decreasing in  $\tau$ .

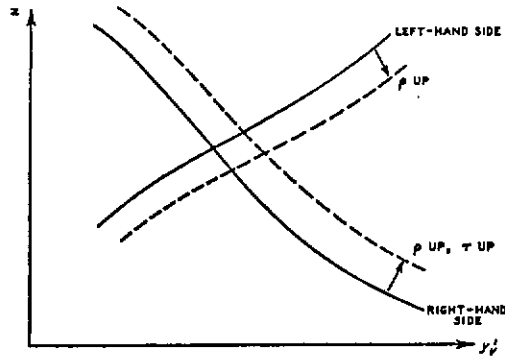


Figure 3

The function  $\rho''$  is graphed in Figure 2. The curves  $\rho'$  and  $\rho''$  in Figure 2 can intersect at most once, and hence, with equation (11), give a *unique* solution for  $y'$ ,  $y'_s$ , and  $\rho$  in terms of  $k$ ,  $\theta$ , and  $\tau$ . Hence, the original system (1)–(10) has a unique solution, provided it exists.

Let  $y'_s = \hat{y}'_s(k, \theta, \tau)$ ;  $y' = \hat{y}'(k, \theta, \tau)$ ; and  $\rho = \hat{\rho}(k, \theta, \tau)$  denote the solutions derived from (11)–(13) above. Then, from Figure 2, we see that  $y'_s$  is an *increasing* function of both policy instruments  $\theta$  and  $\tau$ . Since, from (11),  $y'$  is strictly increasing in  $y'_s$  except at factor intensity reversals, it follows that  $\hat{y}'$  is also *increasing* in the policy instruments. We now examine  $\rho = \hat{\rho}(k, \theta, \tau)$ . From Figure 2,  $\hat{\rho}$  is an *increasing* function of  $\theta$ . A more involved argument will show that  $\hat{\rho}$  is an *increasing* function of  $\tau$ . An implication of the concavity of the consumption function in  $(y^{d'}, a')$  is

$$(15) \quad C(\rho, y^{d'}, a') - C(\rho, 0, 0) \geq C_2(\rho, y^{d'}, a')y^{d'} + C_3(\rho, y^{d'}, a')a'$$

or

$$1 \geq \epsilon(c'; y^{d'}) + \epsilon(c', a'),$$

where  $\epsilon(f, x)$  is the elasticity of  $f$  with respect to  $x$ . That is, the sum of the elasticities of consumption with respect to disposable income and real assets cannot exceed one. Now, differentiating the system (11)–(13) yields

$$\begin{bmatrix} -g_2 & 1 & 0 \\ h_2(1-\tau)(\theta\lambda'_2 - 1) & \theta\lambda'_1 & 0 \\ -h_2(1-\tau)[C_2 + C_3\lambda'_2] + Z'_1 & C_1 + C_3\lambda'_1 + Z'_1 & 0 \end{bmatrix} \begin{bmatrix} dy'_s \\ dy' \\ d\rho \end{bmatrix} = \begin{bmatrix} 0 \\ y'(\theta\lambda'_2 - 1) \\ y'(C_2 + C_3\lambda'_2) \end{bmatrix} d\tau.$$

The determinant of this matrix is negative, and the numerator in the Cramer's rule solution for  $\partial\rho/\partial\tau$  is

$$(16) \quad \text{numerator} = g_2 Z'_1 [\theta\lambda'_2 - 1] - h_2 y' (C_2 + C_3\lambda'_2 + \theta\lambda'_2 - 1).$$

From (14),  $\theta\lambda'_2 - 1 \leq -h/y^{d'}$ . Further,  $\lambda'_2 \leq \frac{\lambda'}{y^{d'}} \leq \frac{a'}{y^{d'}}$  and (15) yields  $C_2 + C_3\lambda'_2 \leq [C_2 y^{d'} + C_3 a']/y^{d'} \leq C/y^{d'} < h/y^{d'}$ . Hence, both terms of the numerator in (16) are negative.

In general, the impact of the policy instruments  $\theta$  and  $\tau$  on the equilibrium investment level  $\hat{z}'$  is not determinate. However, let us now assume, as Professor Uzawa does, that the liquidity preference function and consumption are perfectly interest inelastic (i.e.,  $\lambda'_1 = 0$  and  $C_1 = 0$ ). Then  $z' = \hat{z}'(k, \theta, \tau)$  can be shown to be a strictly *decreasing* function of  $\theta$  and a strictly *increasing* function of  $\tau$ .

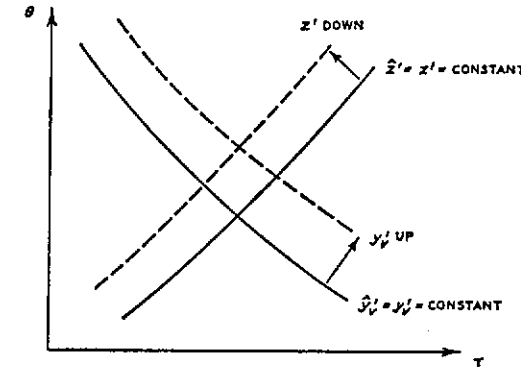


Figure 4

As illustrated in Figure 4, the values of the policy instruments  $\theta$  and  $\tau$  are then uniquely determined as functions of the predetermined capital stock  $k$  and a planner's desired levels of the target variables  $z'$  and  $y'_s$ :  $\theta = \hat{\theta}(k, y'_s, z')$  and  $\tau = \hat{\tau}(k, y'_s, z')$ , where  $\hat{\theta}$  is *increasing* in  $y'_s$  and *decreasing* in  $z'$ ; and  $\hat{\tau}$  is *increasing* in  $y'_s$  and in  $z'$ .

The problem of atemporal and/or intertemporal maximization of a planner's preference function (social utility function) by choice of the mix of public and private goods and of the rate of investment can now be analyzed directly in terms of the target variables  $y'_s$  and  $z'$ , with the  $\hat{\theta}$  and  $\hat{\tau}$  functions specifying the appropriate fiscal policy for any set of targets.

#### IV. CONCLUSIONS

Professor Uzawa's result illustrates the common proposition that policy analysis can be couched directly in terms of target variables (public goods/man, in-

vestment/man) provided their number does not exceed the number of independent instrument variables (rate of increase of real money supply, income tax rate). A corollary to this observation is that additional flexibility in policy could be obtained by expanding the model to include additional instrument variables. A first step might be the introduction of government bonds, allowing the government some freedom in the control of the interest rate via its portfolio decision on the mix of new issues of interest-bearing and non-interest-bearing debt. This step should give a somewhat closer approximation to the financial instruments actually available to governments. At the next level of complexity, it would be desirable to include in the model a foreign trade sector, with foreign debt and the exchange rate as instruments, and additional domestic instrument variables: differential tax rates on wage and non-wage income, direct transfers to the private sector, direct government investment, and private investment credits. The gain in flexibility in choice of instruments in a more complex model would be partially offset by the addition of new target variables, including possibly income distribution, foreign debt and exchange reserves, employment levels (if the current condition of competitive factor markets is relaxed), and the rate of inflation.

Finally, Professor Uzawa's paper suggests (and leaves open) a number of interesting questions. What is the impact of government policy changes in the presence of a more complex expectations structure than the "rain today, rain tomorrow" expectations implicit in the current model? What is the behavior of the economy when the current assumptions of perfectly clearing markets are relaxed (particularly, the labor market)? What are the qualitative characteristics of optimal fiscal and monetary policies? In the steady state growth of the Golden Age, what are the levels of the savings rate, the income tax rate, the rate of inflation, and the rate of increase of the money supply? How do these values compare with current ranges of political feasibility?

## PART II

### DEVELOPMENT PLANNING AND PROGRAMMING